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Galerkin alternating-direction method for a kind of three-dimensional nonlinear hyperbolic problems

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ABSTRACT

A kind of three-dimensional nonlinear hyperbolic equation is firstly transformed into a first-order system of equations, then the Galerkin alternating-direction procedure for the system is derived. The error estimates of the procedure in H^1 norm and L^2 norm, respectively, are obtained by using the theory and techniques of priori estimate of differential equations. The numerical experiment is also given to support the theoretical analysis. Comparing the results of numerical example with the theoretical analysis, they are coincided.

1. Introduction

Galerkin alternating-direction procedure, which was first proposed by Douglas and Dupont [1], is of interest because it can transform multi-dimensional problems into a series of one-dimensional problems and there is a significant reduction in the computing time and storage requirements. Dendy, Fairweather, etc. have studied the Galerkin alternating-direction procedure for hyperbolic equations on rectangular regions, see [2–7] for instance. [1] discussed the simplest hyperbolic equation and derived H_0^1 norm error estimate. [2] generalized the method formulated in [1]. A new method was put forward in [3] by transforming the second-order hyperbolic equation into a first-order system of equations and the H_0^1 norm and the L^2 norm error estimates were derived. For this method, although the variable time steps can be employed and the initial conditions can be determined in a more natural way, it investigated a special kind of hyperbolic equation with separable coefficients. [4,5,7] also applied this method to some kinds of hyperbolic equations. However, [4,5] only gave the finite element method and did not discuss the Galerkin alternating-direction procedure. [7] studied the Galerkin alternating-direction method for a kind of second-order quasi-linear hyperbolic equation

$$u_{tt} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(\boldsymbol{x})\frac{\partial u}{\partial x_j}\right) = f(\boldsymbol{x}, t)$$

and no numerical example was given.

Consider a general form of nonlinear hyperbolic problem:

$$q(\boldsymbol{x})u_{tt} - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}(\boldsymbol{x}, u)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{3} b_i(\boldsymbol{x}, u)\frac{\partial u}{\partial x_i} = f(\boldsymbol{x}, t, u), \quad (\boldsymbol{x}, t) \in \Omega \times J, \tag{1.1}$$

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$$u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times J, \tag{1.2}$$
$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad x \in \Omega, \tag{1.3}$$
$$u_t(\boldsymbol{x}, 0) = u_1(\boldsymbol{x}), \quad x \in \Omega, \tag{1.4}$$

where $\boldsymbol{x} = (x, y, z) = (x_1, x_2, x_3)$, $J = [0, T]$, $\Omega : [a, b] \times [c, d] \times [e, f]$ is a cubic region in R^3. Coefficients $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$ and for $\forall(\xi_1, \xi_2, \xi_3) \in \Omega$, there exist a positive constant $\alpha > 0$ and M such that

$$\sum_{i,j=1}^{3} a_{ij}\xi_i\xi_j \geq \alpha(\xi_1^2 + \xi_2^2 + \xi_3^2)$$

and $\left|\frac{\partial a_{ij}(\boldsymbol{x},u)}{\partial u}\right| \leq M$. $q(\boldsymbol{x})$ satisfies $0 < q_* < q(\boldsymbol{x}) < q^*$. Functions $b(\boldsymbol{x}, u)$ and $f(\boldsymbol{x}, t, u)$ are Lipschitz continuous in ε_0-neighborhood of the solution. That is, when $\left|\varepsilon_j\right| \leq \varepsilon_0$ $(j = 1, 2, 3, 4)$ there exist positive constant M, such that

$$\begin{aligned} &|b_i(u(\boldsymbol{x}, t) + \varepsilon_1) - b_i(u(\boldsymbol{x}, t) + \varepsilon_2)| \leq M\,|\varepsilon_1 - \varepsilon_2|\,, \quad i = 1, 2, 3 \\ &|f(u(\boldsymbol{x}, t) + \varepsilon_3) - f(u(\boldsymbol{x}, t) + \varepsilon_4)| \leq M\,|\varepsilon_3 - \varepsilon_4|\,. \end{aligned} \tag{1.5}$$

In this paper, based on the method proposed in [3], a Galerkin alternating-direction scheme for the general form of three-dimensional nonlinear hyperbolic problem (1.1)–(1.4) is proposed. The H^1 norm and the L^2 norm error estimates of the procedure are derived by using the theory and techniques of priori estimate of differential equations. The most important contribution is that the numerical example for this method is given. This has not been found in past research. The result of this paper is important for the theoretical analysis and practical computation of nonlinear vibration problems (see [8] for example).

The outline of the paper is as follows. In Section 2, some notations and assumptions are introduced. In Section 3, the hyperbolic equation is transformed into a system of equations and the Galerkin alternating-direction scheme is proposed. In Sections 4 and 5, the H^1 norm and the L^2 norm error estimates of the procedure are derived, respectively. In Section 6, the matrix problems of the scheme are given. And finally in Section 7, a numerical example is given to show that the method proposed in this paper is indeed a highly efficient one for science-engineering computation.

In this paper, C is positive constant and ε is small positive constant.

2. Preliminary and notations

For a positive integer s, $H^s(\Omega)$, which norm is denoted as $\|\cdot\|_s$, is the Sobolev space $W_2^s(\Omega)$ in the norm

$$\|v\|_s = \left(\sum_{0\leq\alpha_1+\alpha_2+\alpha_3\leq s} \left\|\frac{\partial^{\alpha_1+\alpha_2+\alpha_3} v}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\right\|^2\right)^{\frac{1}{2}}.$$

Let $(f, g) = \int_\Omega fg\mathrm{d}x$, and $\|f\|$ denote the $L^2(\Omega)$ inner product and norm. $H_0^1(\Omega)$ is the closure of $C_0^\infty(\bar{\Omega})$ with respect to the norm $\|\cdot\|_1$, and define an equivalent norm on $H_0^1(\Omega)$ as

$$\|v\|_{H_0^1} = \|\nabla v\| = \left(\left\|\frac{\partial v}{\partial x_1}\right\|^2 + \left\|\frac{\partial v}{\partial x_2}\right\|^2 + \left\|\frac{\partial v}{\partial x_3}\right\|^2\right)^{\frac{1}{2}}.$$

If X is a normed space with the norm $\|\cdot\|_X$ and a map $v : [0, T] \to X$, define

$$\|v\|_{L^2(X)} = \left(\int_0^T \|v(t)\|_X^2\,\mathrm{d}t\right)^{\frac{1}{2}},$$
$$\|v\|_{L^\infty(X)} = \sup_{0\leq t\leq T} \|v(t)\|_X\,.$$

Denote $D_1 = \frac{\partial^2}{\partial x_1\partial x_2}$, $D_2 = \frac{\partial^2}{\partial x_1\partial x_3}$, $D_3 = \frac{\partial^2}{\partial x_2\partial x_3}$, $D_4 = \frac{\partial^3}{\partial x_1\partial x_2\partial x_3}$, and $Z = \{\phi|\phi, \phi_{x_1}, \phi_{x_2}, \phi_{x_3}, D_i\phi\ (i = 1, 2, 3, 4) \in L^2(\Omega)\}$. Let $S_{h,r}(\Omega)(r \geq 2)$ be the finite-dimensional subspaces of $H_0^1(\Omega)$ and satisfy

$$S_{h,r} \in Z \cap H_0^1, \tag{2.1a}$$

$$\inf_{\chi\in S_{h,r}} \left(\sum_{m=0}^{3} h^m \sum_{\substack{i,j,k=0,1\\ i+j+k=m}} \left\|\frac{\partial^m(\phi - \chi)}{\partial x_1^i\partial x_2^j\partial x_3^k}\right\|\right) \leq Mh^s\,\|\phi\|_s\,, \quad \phi \in H^s(\Omega) \cap Z, 2 \leq s \leq r. \tag{2.1b}$$

Define $W : (0, T] \to S_{h,r}(\Omega)$ as the weighted H^1 projection of the solution $u(\boldsymbol{x}, t)$ of (1.1):

$$\sum_{i,j=1}^{3}\left(a_{ij}(\boldsymbol{x}, u)\frac{\partial(u-W)}{\partial x_j}, \frac{\partial V}{\partial x_i}\right) + \sum_{i=1}^{3} b_i(\boldsymbol{x}, u)\left(\frac{\partial(u-W)}{\partial x_i}, V\right) + (\mu(u-W), V) = 0, \quad V \in S_{h,r}, \tag{2.2}$$

where μ is a positive constant (see [9] for instance). From (1.1) and (2.2), W satisfies

$$(q(\boldsymbol{x})u_{tt}, V) + (a(u)\nabla W, \nabla V) + (b(u)\nabla W, V) + (\mu(W-u), V) = (f(u), V), \quad V \in S_{h,r}. \tag{2.3}$$

In convenience, the following notations will be used. Let U^n be the Galerkin alternating-direction solution at the n-th time step and for $n = 1, \ldots, M = \frac{T}{\Delta t}$, define

$$t^n = n\Delta t, \qquad \varphi^n = \varphi(\boldsymbol{x}, t^n), \qquad \varphi^{n+\frac{1}{2}} = \frac{\varphi^{n+1}+\varphi^n}{2}, \qquad d_t\varphi^n = \frac{\varphi^{n+1}-\varphi^n}{\Delta t},$$

$$(a(u)\nabla u, \nabla v) = \sum_{i,j=1}^{3}\left(a_{ij}(u)\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}\right), \qquad (b(u)\nabla u, v) = \sum_{i=1}^{3}\left(b_i(u)\frac{\partial u}{\partial x_j}, v\right),$$

and three lemmas are firstly introduced.

Lemma 1 ([9]). *Let $\eta = u - W$, for $0 \le k \le 2$, $p = 2, \infty$, $j = 0, 1$, there exists a constant C, independent of h, such that*

$$\left\|\frac{\partial^k \eta}{\partial t^k}\right\|_{L^p(H^j)} \le Ch^{s-j}\left\|\frac{\partial^k u}{\partial t^k}\right\|_{L^p(H^s)}, \quad 1 \le s \le r \tag{2.4}$$

$$\|\nabla W\|_{L^\infty(J;L^\infty(\Omega))} + \|\nabla W_t\|_{L^\infty(J;L^\infty(\Omega))} + \|W_t\|_{L^\infty(J;L^\infty(\Omega))} \le C. \tag{2.5}$$

Lemma 2 ([10]). *Let D denote Δ, ∇ or D_i $(i = 1, 2, 3, 4)$, then for appropriate v,*

$$\Delta t\sum_{n=0}^{M-1}\left\|Dd_t v^n\right\|^2 \le \left\|D\frac{\partial v}{\partial t}\right\|^2_{L^2(L^2)}. \tag{2.6}$$

Lemma 3 ([2]). *If D denote the operators d_t, $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial t^2}$, then for $r \ge 3$,*

$$\left\|\frac{\partial^2(D\eta^n)}{\partial x_i\partial x_j}\right\| \le Ch^{r-2}\|Du\|_r + Ch^{-2}\left\|D\eta^n\right\|, \quad i, j = 1, 2, 3, \tag{2.7}$$

$$\left\|\frac{\partial^3(D\eta^n)}{\partial x_1\partial x_2\partial x_3}\right\| \le Ch^{r-3}\|Du\|_r + Ch^{-3}\left\|D\eta^n\right\|. \tag{2.8}$$

3. Formulation of Galerkin alternating-direction methods

Let $\frac{\partial u}{\partial t} = \phi$, (1.1)–(1.4) can be written as

$$\begin{cases} q(\boldsymbol{x})\dfrac{\partial\phi}{\partial t} - \displaystyle\sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\left(a_{ij}(\boldsymbol{x}, u)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{3} b_i(\boldsymbol{x}, u)\frac{\partial u}{\partial x_i} = f(\boldsymbol{x}, t, u), \\ \dfrac{\partial u}{\partial t} = \phi, \\ u(\boldsymbol{x}, 0) = u_0, \\ \phi(\boldsymbol{x}, 0) = u_1. \end{cases} \tag{3.1}$$

The weak form of the problem (3.1) is:

$$\begin{cases} \left(q(\boldsymbol{x})\dfrac{\partial\phi}{\partial t}, V\right) + (a(u)\nabla u, \nabla V) + (b(u)\nabla u, V) = (f(u), V), \quad V \in S_{h,r}, \\ \dfrac{\partial u}{\partial t} = \phi, \\ (u(\boldsymbol{x}, 0), v) = (u_0, v), \\ (\phi(\boldsymbol{x}, 0), v) = (u_1, v). \end{cases} \tag{3.2}$$

Assume that the Galerkin approximation to the solution u is a differentiable map $U: [0, T] \to S_{h,r}$ such that

$$(q(\boldsymbol{x})d_t\Phi^n, V) + \lambda\Delta t(q(\boldsymbol{x})\nabla d_tU^n, \nabla V) + (a(U^n)\nabla U^n, \nabla V)$$
$$= (f(U^n), V) - (b(U^n)\nabla U^n, V), \quad V \in S_{h,r}, n = 0, 1, 2, \ldots \tag{3.3a}$$

$$d_tU^n = \Phi^{n+\frac{1}{2}}, \quad n = 0, 1, 2, \ldots \tag{3.3b}$$

where $\lambda > \frac{1}{2}(\max_{x\in\Omega} \|A(\boldsymbol{x}, U^n)\| /q_*)$, $\|A(\boldsymbol{x}, U^n)\|$ is the norm of the matrix $A(\boldsymbol{x}, U^n) = \{a_{ij}(\boldsymbol{x}, U^n)\}$ and the differentiable map $\Phi : [0, T] \to S_{h,r}$ is an approximation to ϕ. Let $E^{n+1} = \Phi^{n+1} - \Phi^n$, then $\Phi^{n+\frac{1}{2}} = \frac{E^{n+1}}{2} + \Phi^n$, (3.3a) and (3.3b) can be transformed into

$$\left(q(\boldsymbol{x})\frac{E^{n+1}}{\Delta t}, V\right) + \frac{1}{2}\lambda(\Delta t)^2\left(q(\boldsymbol{x})\nabla\frac{E^{n+1}}{\Delta t}, \nabla V\right)$$
$$= (f(U^n), V) - (a(U^n)\nabla U^n, \nabla V) - (b(U^n)\nabla U^n, V) - \lambda\Delta t(q(\boldsymbol{x})\nabla\Phi^n, \nabla V) \quad V \in S_{h,r}, n = 0, 1, 2, \ldots \tag{3.4a}$$

$$U^{n+1} = U^n + \Delta t\Phi^n + \frac{\Delta t}{2}E^{n+1}, \quad n = 0, 1, 2, \ldots. \tag{3.4b}$$

According to [1], the Galerkin alternating-direction scheme of (3.1) can be defined as:

$$(q(\boldsymbol{x})E^{n+1}, V) + \frac{1}{2}\lambda(\Delta t)^2(q(\boldsymbol{x})\nabla E^{n+1}, \nabla V) + \frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(q(\boldsymbol{x})D_iE^{n+1}, D_iV) + \frac{1}{8}\lambda^3(\Delta t)^6(q(\boldsymbol{x})D_4E^{n+1}, D_4V)$$
$$= \Delta t[(f(U^n), V) - (a(U^n)\nabla U^n, \nabla V) - (b(U^n)\nabla U^n, V) - \lambda\Delta t(q(\boldsymbol{x})\nabla\Phi^n, \nabla V)], \quad V \in S_{h,r}, n = 0, 1, 2, \ldots \tag{3.5a}$$

$$U^{n+1} = U^n + \Delta t\Phi^n + \frac{\Delta t}{2}E^{n+1}, \quad n = 0, 1, 2, \ldots. \tag{3.5b}$$

The initial values W^0, $\left(\frac{\partial W}{\partial t}\right)^0$ can be found from

$$\sum_{i,j=1}^{3}\left(a_{ij}(\boldsymbol{x}, u_0)\frac{\partial(u_0 - W^0)}{\partial x_j}, \frac{\partial V}{\partial x_i}\right) + \sum_{i=1}^{3}b_i(\boldsymbol{x}, u_0)\left(\frac{\partial(u_0 - W^0)}{\partial x_i}, V\right) + ((u_0 - W^0), V) = 0, \tag{3.6}$$

$$\sum_{i,j=1}^{3}\left(a_{ij}(\boldsymbol{x}, u_1)\frac{\partial}{\partial x_j}\left(u_1 - \left(\frac{\partial W}{\partial t}\right)^0\right), \frac{\partial V}{\partial x_i}\right)$$
$$+ \sum_{i=1}^{3}b_i(\boldsymbol{x}, u_1)\left(\frac{\partial}{\partial x_i}\left(u_1 - \left(\frac{\partial W}{\partial t}\right)^0\right), V\right) + \left(\left(u_1 - \left(\frac{\partial W}{\partial t}\right)^0\right), V\right) = 0, \tag{3.7}$$

and let $U^0 = W^0$, $\Phi^0 = \left(\frac{\partial W}{\partial t}\right)^0$.

4. H^1-error estimates

The equivalent form of (3.5) is:

$$(q(\boldsymbol{x})d_t\Phi^n, V) + \lambda\Delta t(q(\boldsymbol{x})\nabla d_tU^n, \nabla V) + \frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(q(\boldsymbol{x})D_i(d_t\Phi^n), D_iV) + \frac{1}{8}\lambda^3(\Delta t)^6(q(\boldsymbol{x})D_4(d_t\Phi^n), D_4V)$$
$$= (f(U^n), V) - (a(U^n)\nabla U^n, \nabla V) - (b(U^n)\nabla U^n, V), \tag{4.1a}$$

$$d_tU^n = \Phi^{n+\frac{1}{2}}, \quad n = 0, 1, 2, \ldots. \tag{4.1b}$$

Discretizing (3.2) yields:

$$\left(\left(\frac{\partial\phi}{\partial t}\right)^n, V\right) + (a(u^n)\nabla u^n, \nabla V) + (b(u^n)\nabla u^n, V) = (f(u^n), V), \quad V \in S_{h,r}(\Omega), \tag{4.2a}$$

$$\left(\frac{\partial u}{\partial t}\right)^{n+\frac{1}{2}} = \phi^{n+\frac{1}{2}}. \tag{4.2b}$$

Let $\xi^n = U^n - W^n$, $\eta^n = u^n - W^n$, $\hat{\xi}^n = \Phi^n - \left(\frac{\partial W}{\partial t}\right)^n$, $\hat{\eta}^n = \phi^n - \left(\frac{\partial W}{\partial t}\right)^n$, then

$$\xi^n - \eta^n = U^n - u^n, \qquad \hat{\xi}^n - \hat{\eta}^n = \Phi^n - \phi^n.$$

Subtracting (4.2) from (4.1) and using (2.2) lead to

$$(q(\boldsymbol{x})d_t\hat{\xi}^n, V) + \lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^n, \nabla V) + \frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(q(\boldsymbol{x})D_i(d_t\hat{\xi}^n), D_iV)$$
$$+\frac{1}{8}\lambda^3(\Delta t)^6(q(\boldsymbol{x})D_4(d_t\hat{\xi}^n), D_4V) + (a(U^n)\nabla\xi^n, \nabla V)$$
$$= \left(q(\boldsymbol{x})\left(\left(\frac{\partial\phi}{\partial t}\right)^n - d_t\phi^n + d_t\hat{\eta}^n\right) - \mu\eta^n + f(U^n) - f(u^n) + (b(u^n) - b(U^n))\nabla W^n - b(U^n)\nabla\xi^n, V\right)$$
$$-\lambda\Delta t(q(\boldsymbol{x})\nabla d_t(u^n - \eta^n), \nabla V) + \frac{\lambda^2}{4}(\Delta t)^4\sum_{i=1}^{3}(q(\boldsymbol{x})D_i(d_t\hat{\eta}^n - d_t\phi^n), D_iV)$$
$$+\frac{\lambda^3}{8}(\Delta t)^6(q(\boldsymbol{x})D_4(d_t\hat{\eta}^n - d_t\phi^n), D_4V) + ((a(u^n) - a(U^n))\nabla W^n, \nabla V), \tag{4.3a}$$
$$d_t\xi^n = \hat{\xi}^{n+\frac{1}{2}} + \rho^n, \tag{4.3b}$$

where $\rho^n = d_t\eta^n - \hat{\eta}^{n+\frac{1}{2}} + (\frac{\partial u}{\partial t})^{n+\frac{1}{2}} - d_tu^n$.

Choosing $V = \hat{\xi}^{n+\frac{1}{2}} = d_t\xi^n - \rho^n$, and since $\xi^n = \xi^{n+\frac{1}{2}} - \frac{\Delta t}{2}d_t\xi^n$, hence

$$\lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^n, \nabla V) + (a(U^n)\nabla\xi^n, \nabla V)$$
$$= \lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^n, \nabla(d_t\xi^n - \rho^n)) + \left(a(U^n)\nabla\left(\xi^{n+\frac{1}{2}} - \frac{\Delta t}{2}d_t\xi^n\right), \nabla d_t\xi^n\right) - (a(U^n)\nabla\xi^n, \nabla\rho^n)$$
$$= \lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^n, \nabla d_t\xi^n) - \lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^n, \nabla\rho^n)$$
$$-\frac{\Delta t}{2}(a(U^n)\nabla d_t\xi^n, \nabla d_t\xi^n) + (a(U^n)\nabla\xi^{n+\frac{1}{2}}, \nabla d_t\xi^n) - (a(U^n)\nabla\xi^n, \nabla\rho^n)$$
$$= \frac{1}{2}\Delta t((2\lambda q(\boldsymbol{x}) - a(\boldsymbol{x}, U^n))\nabla d_t\xi^n, \nabla d_t\xi^n)$$
$$+(a(U^n)\nabla\xi^{n+\frac{1}{2}}, \nabla d_t\xi^n) - \lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^n, \nabla\rho^n) - (a(U^n)\nabla\xi^n, \nabla\rho^n),$$

also

$$\begin{aligned}(a(U^n)\nabla\xi^{n+\frac{1}{2}}, \nabla d_t\xi^n) &= \left(a(U^n)\nabla\left(\frac{\xi^n + \xi^{n+1}}{2}\right), \nabla\left(\frac{\xi^{n+1} - \xi^n}{\Delta t}\right)\right)\\
&= \frac{1}{2\Delta t}\left\{(a(U^n)\nabla\xi^{n+1}, \nabla\xi^{n+1}) - (a(U^n)\nabla\xi^n, \nabla\xi^n)\right\}\\
&= \frac{1}{2\Delta t}\{(a(U^n)\nabla\xi^{n+1}, \nabla\xi^{n+1}) - (a(U^{n-1})\nabla\xi^n, \nabla\xi^n)\} + \frac{1}{2\Delta t}((a(U^{n-1}) - a(U^n))\nabla\xi^n, \nabla\xi^n)\\
&= \frac{1}{2}d_t(a(U^{n-1})\nabla\xi^n, \nabla\xi^n) + \frac{1}{2\Delta t}\cdot\left(\frac{\partial a}{\partial u}\cdot(-d_tU^{n-1})\nabla\xi^n, \nabla\xi^n\right)\Delta t\\
&= \frac{1}{2}d_t(a(U^{n-1})\nabla\xi^n, \nabla\xi^n) + \frac{1}{2}\left(\frac{\partial a}{\partial u}\cdot(-d_tU^{n-1})\nabla\xi^n, \nabla\xi^n\right),\end{aligned}$$

thus

$$\lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^n, \nabla V) + (a(U^n)\nabla\xi^n, \nabla V)$$
$$\geq \hat{q}\Delta t\left\|d_t\xi^n\right\|^2_{H_0^1} + \frac{1}{2}d_t(a(U^{n-1})\nabla\xi^n, \nabla\xi^n)$$
$$+\frac{1}{2}\left(\frac{\partial a}{\partial u}\cdot(-d_tU^{n-1})\nabla\xi^n, \nabla\xi^n\right) - \varepsilon\Delta t\left\|\nabla d_t\xi^n\right\|^2 - C\left(\left\|\xi^n\right\|^2_{H_0^1} + \left\|\nabla\rho^n\right\|^2\right)$$
$$\geq (\hat{q} - \varepsilon)\Delta t\left\|d_t\xi^n\right\|^2_{H_0^1} + \frac{c_*}{2}d_t(\nabla\xi^n, \nabla\xi^n) + \frac{1}{2}\left(\frac{\partial a}{\partial u}\cdot(-d_tU^{n-1})\nabla\xi^n, \nabla\xi^n\right) - C(\left\|\xi^n\right\|^2_{H_0^1} + \left\|\nabla\rho^n\right\|^2), \tag{4.4}$$

where $\hat{q} = \frac{1}{2}\min(\|2\lambda q(\boldsymbol{x})I - A(\boldsymbol{x}, U^n)\|)$, I is a 3 by 3 identity matrix. A is a 3 by 3 matrix, which components are $a_{ij}(U^n)$. $\|2\lambda q(\boldsymbol{x})I - A(\boldsymbol{x}, U^n)\|$ is the norm of the matrix $2\lambda q(\boldsymbol{x})I - A(\boldsymbol{x}, U^n)$. $\hat{q} - \varepsilon > 0$ is true provided that $\lambda > \frac{1}{2}(\max_{\boldsymbol{x}\in\Omega}\|A(\boldsymbol{x}, U^n)\|/q_*)$.

Because $b(\boldsymbol{x}, u)$ and $f(\boldsymbol{x}, t, u)$ satisfy (1.5), hence

$$\left|f(U^n) - f(u^n)\right| = \left|f(u^n + U^n - u^n) - f(u^n)\right| \leq M\left|\varepsilon - 0\right| = M\varepsilon$$

and

$$\left|b(U^n)-b(u^n)\right| = \left|b(u^n+U^n-u^n)-b(u^n)\right| \le M\,|\varepsilon-0| = M\varepsilon$$

are true provided that

$$\left|U^n-u^n\right| = \varepsilon \le \varepsilon_0.$$

Thus, we introduce the induction hypothesis:

$$\max_{0\le n\le M}\left\|U^n-u^n\right\|_{L^\infty} \le \varepsilon_0, \tag{4.5}$$

where ε_0 is a positive number. From (2.5),

$$\left|((a(u^n)-a(U^n))\nabla W^n,\nabla V)\right| \le C\left|\nabla((a(u^n)-a(U^n)),V)\right| \le C\left(\left\|\nabla\xi^n\right\|^2+\left\|\nabla\eta^n\right\|^2\right)+\left\|\hat{\xi}^{n+\frac{1}{2}}\right\|^2. \tag{4.6}$$

Using Schwartz inequality, $ab\le\frac{1}{2}(a^2+b^2)$ and (4.4)–(4.6), (4.3) can be transformed into:

$$\begin{aligned}
&(d_t\hat{\xi}^n,V)+(\hat{q}-\varepsilon)\Delta t\left\|d_t\xi^n\right\|^2_{H_0^1}+\frac{C}{2}d_t(\nabla\xi^n,\nabla\xi^n)+\frac{\lambda^2}{4}(\Delta t)^4\sum_{i=1}^{3}(D_i(d_t\hat{\xi}^n),D_iV)+\frac{\lambda^3}{8}(\Delta t)^6(D_4(d_t\hat{\xi}^n),D_4V)\\
&\le C\left[\left\|\left(\frac{\partial\phi}{\partial t}\right)^n-d_t\phi^n+d_t\hat{\eta}^n\right\|^2+\left\|\xi^n\right\|_1^2+\left\|\eta^n\right\|^2+\left\|\hat{\xi}^{n+\frac{1}{2}}\right\|^2+(\Delta t)^2\left\|\Delta d_t(u^n-\eta^n)\right\|^2\right.\\
&\quad+C\left\|d_tU^{n-1}\right\|_{L^\infty}\left\|\nabla\xi^n\right\|^2+(\Delta t)^4\sum_{i=1}^{3}\left\|D_i(d_t\hat{\eta}^n-d_t\phi^n)\right\|^2+(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\hat{\xi}^{n+\frac{1}{2}}\right\|^2\\
&\quad\left.+(\Delta t)^6\left\|D_4(d_t\hat{\eta}^n-d_t\phi^n)\right\|^2+(\Delta t)^6\left\|D_4\hat{\xi}^{n+\frac{1}{2}}\right\|^2+\left\|\xi^n\right\|^2_{H_0^1}+\left\|\nabla\rho^n\right\|^2\right].
\end{aligned} \tag{4.7}$$

Since

$$\begin{aligned}
&(d_t\hat{\xi}^n,\hat{\xi}^{n+\frac{1}{2}})=\frac{1}{2\Delta t}[(\hat{\xi}^{n+1},\hat{\xi}^{n+1})-(\hat{\xi}^n,\hat{\xi}^n)]=\frac{1}{2}d_t(\hat{\xi}^n,\hat{\xi}^n),\\
&(D_i(d_t\hat{\xi}^n),D_i(\hat{\xi}^{n+\frac{1}{2}}))=\frac{1}{2}d_t(D_i\hat{\xi}^n,D_i\hat{\xi}^n),\quad i=1,2,3,4,
\end{aligned}$$

then (4.7) becomes

$$\begin{aligned}
&d_t\left[(\hat{\xi}^n,\hat{\xi}^n)+\frac{C}{2}(\nabla\xi^n,\nabla\xi^n)+\frac{\lambda^2}{4}(\Delta t)^4\sum_{i=1}^{3}(D_i\hat{\xi}^n,D_i\hat{\xi}^n)+\frac{\lambda^3}{8}(\Delta t)^6(D_4\hat{\xi}^n,D_4\hat{\xi}^n)\right]+(\hat{q}-\varepsilon)\Delta t\left\|d_t\xi^n\right\|^2_{H_0^1}\\
&\le C\left[\left\|\left(\frac{\partial\phi}{\partial t}\right)^n-d_t\phi^n+d_t\hat{\eta}^n\right\|^2+\left\|\xi^n\right\|_1^2+\left\|\eta^n\right\|^2+\left\|\hat{\xi}^{n+\frac{1}{2}}\right\|^2+(\Delta t)^2\left\|\Delta d_t(u^n-\eta^n)\right\|^2\right.\\
&\quad+C\left\|d_tU^{n-1}\right\|_{L^\infty}\left\|\nabla\xi^n\right\|^2+(\Delta t)^4\sum_{i=1}^{3}\left\|D_i(d_t\hat{\eta}^n-d_t\phi^n)\right\|^2+(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\hat{\xi}^{n+\frac{1}{2}}\right\|^2\\
&\quad\left.+(\Delta t)^6\left\|D_4(d_t\hat{\eta}^n-d_t\phi^n)\right\|^2+(\Delta t)^6\left\|D_4\hat{\xi}^{n+\frac{1}{2}}\right\|^2+\left\|\nabla\rho^n\right\|^2\right].
\end{aligned} \tag{4.8}$$

Multiplying by Δt and summing for $n=0,1,2,\ldots,M-1$ $(0<M\le N)$ yield

$$\begin{aligned}
&\left[(\hat{\xi}^M,\hat{\xi}^M)+\frac{C}{2}(\nabla\xi^M,\nabla\xi^M)+\frac{\lambda^2}{4}(\Delta t)^4\sum_{i=1}^{3}(D_i\hat{\xi}^M,D_i\hat{\xi}^M)+\frac{\lambda^3}{8}(\Delta t)^6(D_4\hat{\xi}^M,D_4\hat{\xi}^M)\right]+(\hat{q}-\varepsilon)(\Delta t)^2\sum_{n=0}^{M-1}\left\|d_t\xi^n\right\|^2_{H_0^1}\\
&\le C\Delta t\sum_{n=0}^{M-1}\left[(\Delta t)^2\left\|\Delta d_t(u^n-\eta^n)\right\|^2+(\Delta t)^4\sum_{I=1}^{3}\left\|D_i(d_t\hat{\eta}^n-d_t\phi^n)\right\|^2+(\Delta t)^6\left\|D_4(d_t\hat{\eta}^n-d_t\phi^n)\right\|^2\right.\\
&\quad\left.+\left\|\left(\frac{\partial\phi}{\partial t}\right)^n-d_t\phi^n+d_t\hat{\eta}^n\right\|^2+\left\|\eta^n\right\|^2+\left\|\nabla\rho^n\right\|^2\right]\\
&\quad+C\Delta t\sum_{n=0}^{M}\left[\left\|\hat{\xi}^n\right\|^2+\left\|\xi^n\right\|_1^2+(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\hat{\xi}^n\right\|^2+(\Delta t)^6\left\|D_4\hat{\xi}^n\right\|^2\right]+C\Delta t\sum_{n=1}^{M-1}\left\|d_tU^{n-1}\right\|_{L^\infty}\left\|\nabla\xi^n\right\|^2\\
&\quad+C\left[\left\|\hat{\xi}^0\right\|^2+\left\|\xi^0\right\|_1^2+(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\hat{\xi}^0\right\|^2+(\Delta t)^6\left\|D_4\hat{\xi}^0\right\|^2\right].
\end{aligned} \tag{4.9}$$

From $\left\|\xi^M\right\|_1^2 = \left\|\nabla\xi^M\right\|^2 + \left\|\xi^M\right\|^2$ and Gronwall inequality, for sufficiently small Δt, if introduce induction hypothesis

$$\sup_{1\le s\le M-1} \Delta t \sum_{n=1}^{s} \left\|d_t U^{n-1}\right\|_{L^\infty} \le C, \tag{4.10}$$

we have

$$\begin{aligned}
&\left[(\hat{\xi}^M,\hat{\xi}^M) + \left\|\xi^M\right\|_1^2 + \frac{\lambda^2}{4}(\Delta t)^4 \sum_{i=1}^{3}(D_i\hat{\xi}^M, D_i\hat{\xi}^M) + \frac{\lambda^3}{8}(\Delta t)^6 (D_4\hat{\xi}^M, D_4\hat{\xi}^M)\right] + (\hat{q}-\varepsilon)(\Delta t)^2 \sum_{n=0}^{M-1}\left\|d_t\xi^n\right\|_{H_0^1}^2 \\
&\le C\Delta t \sum_{n=0}^{M-1}\left[(\Delta t)^2\left\|\Delta d_t(u^n-\eta^n)\right\|^2 + (\Delta t)^4 \sum_{l=1}^{3}\left\|D_i(d_t\hat{\eta}^n - d_t\phi^n)\right\|^2 + (\Delta t)^6\left\|D_4(d_t\hat{\eta}^n - d_t\phi^n)\right\|^2\right. \\
&\quad \left. + \left\|\left(\frac{\partial\phi}{\partial t}\right)^n - d_t\phi^n + d_t\hat{\eta}^n\right\|^2 + \left\|\eta^n\right\|^2 + \left\|\nabla\rho^n\right\|^2\right] \\
&\quad + C\left[\left\|\hat{\xi}^0\right\|^2 + \left\|\xi^0\right\|_1^2 + (\Delta t)^4\sum_{i=1}^{3}\left\|D_i\hat{\xi}^0\right\|^2 + (\Delta t)^6\left\|D_4\hat{\xi}^0\right\|^2\right].
\end{aligned} \tag{4.11}$$

The right-hand side of (4.11) will be estimated term by term.

From Lemma 2, let $D = \Delta$, $v = u, \eta$, then

$$\begin{aligned}
\Delta t\sum_{n=0}^{M-1}(\Delta t)^2\left\|\Delta d_t(u^n-\eta^n)\right\|^2 &\le (\Delta t)^2\left(\Delta t\sum_{n=0}^{M-1}\left\|\Delta d_t u^n\right\|^2 + \Delta t\sum_{n=0}^{M-1}\left\|\Delta d_t\eta^n\right\|^2\right) \\
&\le C(\Delta t)^2\left(\left\|\Delta\frac{\partial u}{\partial t}\right\|_{L^2(L^2)}^2 + \left\|\Delta\frac{\partial\eta}{\partial t}\right\|_{L^2(L^2)}^2\right) \\
&\le C(\Delta t)^2.
\end{aligned} \tag{4.12}$$

From Lemma 2, let $D = D_i$ $(i = 1, 2, 3)$, $v = \hat{\eta}$, then

$$\begin{aligned}
\Delta t\sum_{n=0}^{M-1}\left((\Delta t)^4\sum_{i=1}^{3}\left\|D_i(d_t\hat{\eta}^n - d_t\phi^n)\right\|^2\right) &\le (\Delta t)^4 + (\Delta t)^4\cdot\Delta t\sum_{n=0}^{M-1}\left(\sum_{i=1}^{3}\left\|D_i(d_t\hat{\eta}^n)\right\|^2\right) \\
&\le C\left[(\Delta t)^4 + (\Delta t)^4\sum_{i=1}^{3}\left\|D_i\left(\frac{\partial^2\eta}{\partial t^2}\right)\right\|_{L^2(L^2)}^2\right],
\end{aligned}$$

also from Lemma 3, let $D = \frac{\partial^2}{\partial t^2}$, then for $r \ge 2$, we have

$$\begin{aligned}
\Delta t\sum_{n=0}^{M-1}(\Delta t)^4\left\|D_i(d_t\hat{\eta}^n - d_t\phi^n)\right\|^2 &\le C\left[(\Delta t)^4 + (\Delta t)^4\left\|D_i\left(\frac{\partial^2\eta}{\partial t^2}\right)\right\|_{L^2(L^2)}^2\right] \\
&\le C\left[(\Delta t)^4 + (\Delta t)^4 h^{2r-4}\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(H^r)}^2\right] \le C(\Delta t)^4.
\end{aligned} \tag{4.13}$$

Similarly, when $r \ge 3$, we have

$$\begin{aligned}
\Delta t\sum_{n=0}^{M-1}(\Delta t)^6\left\|D_4(d_t\hat{\eta}^n - d_t\phi^n)\right\|^2 &\le (\Delta t)^6 + (\Delta t)^6\cdot\Delta t\sum_{n=0}^{M-1}\left\|D_4(d_t\hat{\eta}^n)\right\|^2 \\
&\le C\left[(\Delta t)^6 + (\Delta t)^6\left\|D_4\left(\frac{\partial^2\eta}{\partial t^2}\right)\right\|_{L^2(L^2)}^2\right] \\
&\le C\left[(\Delta t)^6 + (\Delta t)^6 h^{2r-6}\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(H^r)}^2\right] \le C(\Delta t)^6.
\end{aligned} \tag{4.14}$$

Let $D=1$, from Lemma 2, and $\left\|\left(\frac{\partial\phi}{\partial t}\right)^n - d_t\phi^n\right\| \le C\Delta t$,

$$\Delta t\sum_{n=0}^{M-1}\left\|\left(\frac{\partial\phi}{\partial t}\right)^n - d_t\phi^n + d_t\hat{\eta}^n\right\|^2 \le C\sum_{n=0}^{M-1}\left[(\Delta t)^2 + \left\|d_t\hat{\eta}^n\right\|^2\right]$$
$$\le C\left[(\Delta t)^2 + \left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(L^2)}\right] \le C[(\Delta t)^2 + h^{2r}]. \tag{4.15}$$

From Lemma 1,

$$\Delta t\sum_{n=0}^{M-1}\left\|\eta^n\right\|^2 \le Ch^{2r}. \tag{4.16}$$

Since

$$\left\|\nabla\left(\left(\frac{\partial u}{\partial t}\right)^{n+\frac{1}{2}} - d_t u^n\right)\right\| \le C(\Delta t)^2,$$

using Lemma 1 and Lemma 2 yields

$$\Delta t\sum_{n=0}^{M-1}\left\|\nabla\rho^n\right\|^2 \le C\left[(\Delta t)^4 + \Delta t\sum_{n=0}^{M-1}\left\|\nabla d_t\eta^n\right\|^2 + \Delta t\sum_{n=0}^{M-1}\left\|\nabla\hat{\eta}^n\right\|^2\right]$$
$$\le C\left[(\Delta t)^4 + \left\|\nabla\frac{\partial\eta}{\partial t}\right\|^2_{L^2(L^2)} + \left\|\nabla\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(L^2)}\right] \le C[(\Delta t)^4 + h^{2r-2}]. \tag{4.17}$$

Combining (4.11)–(4.17), if $\left\|\hat{\xi}^0\right\|^2$, $\left\|\xi^0\right\|_1^2$, $(\Delta t)^4\sum_{i=1}^3\left\|D_i\hat{\xi}^0\right\|^2$ and $(\Delta t)^6\left\|D_4\hat{\xi}^0\right\|^2$ are $O(h^{2r-2})$, it can follow that

$$\left\|\hat{\xi}^M\right\|^2 + \left\|\xi^M\right\|_1^2 + (\Delta t)^4\sum_{i=1}^3\left\|D_i\hat{\xi}^M\right\|^2 + (\Delta t)^6\left\|D_4\hat{\xi}^M\right\|^2 + (\Delta t)^2\sum_{n=0}^{M-1}\left\|d_t\xi^n\right\|^2_{H_0^1} \le C[(\Delta t)^2 + h^{2r-2}]. \tag{4.18}$$

Then by using Lemma 1 and (4.18), yields

$$\max_{0\le n\le N}\left\|U^n - u^n\right\|_1^2 \le C\max_{0\le n\le M}\left[\left\|\xi^n\right\|_1^2 + \left\|\eta^n\right\|_1^2\right] \le C[(\Delta t)^2 + h^{2r-2}]. \tag{4.19}$$

We will check (4.5):

$$\max_{0\le n\le M}\left\|U^n - u^n\right\|_{L^\infty} \le \varepsilon_0.$$

When $n=0$, since we choose that $U^0 = W^0$, from Lemma 1, (4.5) is true.

Assume that when $n = 1, 2, \ldots, M-1$, $\|U^n - u^n\|_{L^\infty} \le \varepsilon_0$ is also true, then for $n = M$, since

$$U^M - u^M = U^M - W^M + W^M - u^M = \xi^M - \eta^M,$$

hence only $\left\|\xi^M\right\|_{L^\infty}$ should be considered. For $\Delta t = O(h^{r-1})$, $r\ge 3$, from (4.18), we have

$$\left\|\xi^M\right\|_{L^\infty} \le Ch^{-\frac{3}{2}}\left\|\xi^M\right\|_{L^2} \le Ch^{-\frac{3}{2}}\left\|\xi^M\right\|_1 \le Ch^{-\frac{3}{2}}(\Delta t + h^{r-1}).$$

For (4.10):

$$\sup_{1\le s\le M-1}\Delta t\sum_{n=1}^{s}\left\|d_t U^{n-1}\right\|_{L^\infty} \le C,$$

since

$$\left\|d_t U^{n-1}\right\|_{L^\infty} \le \left\|d_t\xi^{n-1}\right\|_{L^\infty} + \left\|d_t W^{n-1}\right\|_{L^\infty}.$$

From (2.5), we can only check

$$\sup_{1\le s\le M}\Delta t\sum_{n=1}^{s}\left\|d_t\xi^{n-1}\right\|_{L^\infty} \le C.$$

When $s = 1$, from (4.3) we have $\Delta t \left\| d_t \xi^0 \right\|_{H_0^1} \le C(\Delta t + h^{r-1})$, then for $\Delta t = O(h^{r-1})$, $r \ge 3$,

$$\Delta t \left\| d_t \xi^0 \right\|_{L^\infty} \le Ch^{-\frac{3}{2}} \Delta t \left\| \nabla d_t \xi^0 \right\| \le Ch^{-\frac{3}{2}} (\Delta t + h^{r-1}) \le C.$$

Assume that $\Delta t \sum_{n=1}^{s} \left\| d_t \xi^{n-1} \right\|_{L^\infty} \le C$ holds for $s = 2, 3, \ldots, M-1$, then for $s = M$, from (4.18) leads to

$$\Delta t \sum_{n=1}^{M} \left\| d_t \xi^{n-1} \right\|_{L^\infty} \le \Delta t \sum_{n=1}^{M-1} \left\| d_t \xi^{n-1} \right\|_{L^\infty} + \Delta t \left\| d_t \xi^{M-1} \right\|_{L^\infty} \le C + Ch^{-\frac{3}{2}} \Delta t \left\| \nabla d_t \xi^{M-1} \right\| \le C.$$

So that (4.10) is true for $s = M$. $\square$

Thus, the following theorem can be obtained:

Theorem 1. *Assume that u is the solution to (1.1), let U and W be defined by (3.5) and (2.2), respectively, if $u \in C^4(\bar{\Omega} \times [0, T])$, u, $\frac{\partial u}{\partial t} \in L^\infty(H^r)$, $\frac{\partial^2 u}{\partial t^2} \in L^2(H^r) (r \ge 3)$, $\Delta t = O(h^{r-1})$, $\lambda > \frac{1}{2}(\max_{\boldsymbol{x} \in \Omega} \left\| A(\boldsymbol{x}, U^n) \right\| / q_*)$ and*

$$\left\| \Phi^0 - \left(\frac{\partial W}{\partial t} \right)^0 \right\| + \left\| U^0 - W^0 \right\|_1 + (\Delta t)^2 \sum_{i=1}^{3} \left\| D_i \left[\Phi^0 - \left(\frac{\partial W}{\partial t} \right)^0 \right] \right\|$$
$$+ (\Delta t)^3 \left\| D_4 \left[\Phi^0 - \left(\frac{\partial W}{\partial t} \right)^0 \right] \right\| \le C[\Delta t + h^{r-1}],$$

if h is sufficiently small, and for sufficiently small Δt, there exists a positive constant C such that

$$\max_{0 \le n \le N} \left\| U^n - u^n \right\|_1 \le C[\Delta t + h^{r-1}].$$

5. L^2- error estimates

For any sequence $\{v^n\}_{n=0}^{N}$,

$$v^{n+\frac{1}{2}} = v^0 + \frac{\Delta t}{2} \left[\sum_{k=0}^{n} d_t v^k + \sum_{k=0}^{n-1} d_t v^k \right],$$

let D be an operator, let $v = \hat{\xi}$, and substitute it into (4.3b) yield

$$D(d_t \xi^n) = D\hat{\xi}^0 + \frac{\Delta t}{2} \left[\sum_{k=0}^{n} D(d_t \hat{\xi}^k) + \sum_{k=0}^{n-1} D(d_t \hat{\xi}^k) \right] + D\rho^n.$$

Choosing $D = 1$, D_i $(i = 1, 2, 3, 4)$, and denote

$$I^n = q(\boldsymbol{x}) \left(\left(\frac{\partial \phi}{\partial t} \right)^n - d_t \phi^n + d_t \hat{\eta}^n \right) + f(U^n) - f(u^n) + (b(u^n) - b(U^n)) \nabla W^n - b(U^n) \nabla \xi^n - \mu \eta^n,$$

then from (4.3a) it follows that

$$\begin{aligned}
&(q(\boldsymbol{x}) d_t \xi^n, V) + \frac{1}{4} \lambda^2 (\Delta t)^4 \sum_{i=1}^{3} (q(\boldsymbol{x}) D_i(d_t \xi^n), D_i V) + \frac{1}{8} \lambda^3 (\Delta t)^6 (q(\boldsymbol{x}) D_4(d_t \xi^n), D_4 V) \\
&= \left(q(\boldsymbol{x}) \left(\hat{\xi}^0 + \frac{\Delta t}{2} \left(\sum_{k=0}^{n} d_t \hat{\xi}^k + \sum_{k=0}^{n-1} d_t \hat{\xi}^k \right) + \rho^n \right), V \right) \\
&\quad + \frac{1}{4} \lambda^2 (\Delta t)^4 \sum_{i=1}^{3} \left(q(\boldsymbol{x}) \left(D_i \hat{\xi}^0 + \frac{\Delta t}{2} \left(\sum_{k=0}^{n} D_i(d_t \hat{\xi}^k) + \sum_{k=0}^{n-1} D_i(d_t \hat{\xi}^k) \right) + D_i \rho^n \right), D_i V \right) \\
&\quad + \frac{1}{8} \lambda^3 (\Delta t)^6 \left(q(\boldsymbol{x}) \left(D_4 \hat{\xi}^0 + \frac{\Delta t}{2} \left(\sum_{k=0}^{n} D_4(d_t \hat{\xi}^k) + \sum_{k=0}^{n-1} D_4(d_t \hat{\xi}^k) \right) + D_4 \rho^n \right), D_4 V \right) \\
&= (q(\boldsymbol{x})(\hat{\xi}^0 + \rho^n), V) + \frac{1}{4} \lambda^2 (\Delta t)^4 \sum_{i=1}^{3} (q(\boldsymbol{x}) D_i(\hat{\xi}^0 + \rho^n), D_i V) + \frac{1}{8} \lambda^3 (\Delta t)^6 (q(\boldsymbol{x}) D_4(\hat{\xi}^0 + \rho^n), D_4 V) \\
&\quad + \frac{\Delta t}{2} \sum_{k=0}^{n} \left[(q(\boldsymbol{x}) d_t \hat{\xi}^k, V) + \frac{1}{4} \lambda^2 (\Delta t)^4 \sum_{i=1}^{3} (q(\boldsymbol{x}) D_i(d_t \hat{\xi}^k), D_i V) + \frac{1}{8} \lambda^3 (\Delta t)^6 (q(\boldsymbol{x}) D_4(d_t \hat{\xi}^k), D_4 V) \right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Delta t}{2}\sum_{k=0}^{n-1}\left[(q(\boldsymbol{x})d_t\hat{\xi}^k,V)+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(q(\boldsymbol{x})D_i(d_t\hat{\xi}^k),D_iV)+\frac{1}{8}\lambda^3(\Delta t)^6(q(\boldsymbol{x})D_4(d_t\hat{\xi}^k),D_4V)\right]\\
&=(q(\boldsymbol{x})(\hat{\xi}^0+\rho^n),V)+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(q(\boldsymbol{x})D_i(\hat{\xi}^0+\rho^n),D_iV)+\frac{1}{8}\lambda^3(\Delta t)^6(q(\boldsymbol{x})D_4(\hat{\xi}^0+\rho^n),D_4V)\\
&+\frac{\Delta t}{2}\sum_{k=0}^{n}\Big[-\lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^k,\nabla V)-(a(U^k)\nabla\xi^k,\nabla V)+(I^k,V)-\lambda\Delta t(q(\boldsymbol{x})\nabla d_t(u^k-\eta^k),\nabla V)\\
&+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(q(\boldsymbol{x})D_i(d_t\hat{\eta}^k-d_t\phi^k),D_iV)+\frac{1}{8}\lambda^3(\Delta t)^6(q(\boldsymbol{x})D_4(d_t\hat{\eta}^k-d_t\phi^k),D_4V)\\
&+((a(u^k)-a(U^k))\nabla W^k,\nabla V)\Big]+\frac{\Delta t}{2}\sum_{k=0}^{n-1}\Big[-\lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^k,\nabla V)-(a(U^k)\nabla\xi^k,\nabla V)+(I^k,V)\\
&-\lambda\Delta t(q(\boldsymbol{x})\nabla d_t(u^k-\eta^k),\nabla V)+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(q(\boldsymbol{x})D_i(d_t\hat{\eta}^k-d_t\phi^k),D_iV)\\
&+\frac{1}{8}\lambda^3(\Delta t)^6(q(\boldsymbol{x})D_4(d_t\hat{\eta}^k-d_t\phi^k),D_4V)+((a(u^k)-a(U^k))\nabla W^k,\nabla V)\Big]\\
&=-\left[\lambda\Delta t\left(q(\boldsymbol{x})\nabla\left(\frac{\Delta t}{2}\sum_{k=0}^{n}d_t\xi^k+\frac{\Delta t}{2}\sum_{k=0}^{n-1}d_t\xi^k\right),\nabla V\right)+\frac{\Delta t}{2}\left(\left(\sum_{k=0}^{n}a(U^k)\nabla\xi^k+\sum_{k=0}^{n-1}a(U^k)\nabla\xi^k\right),\nabla V\right)\right]\\
&+\left(q(\boldsymbol{x})(\hat{\xi}^0+\rho^n)+\frac{\Delta t}{2}\sum_{k=0}^{n}I^k+\frac{\Delta t}{2}\sum_{k=0}^{n-1}I^k,V\right)\\
&+\lambda\Delta t\left(q(\boldsymbol{x})\Delta\left(\frac{\Delta t}{2}\sum_{k=0}^{n}d_t(u^k-\eta^k)+\frac{\Delta t}{2}\sum_{k=0}^{n-1}d_t(u^k-\eta^k)\right),V\right)\\
&+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}\left(q(\boldsymbol{x})\left(D_i\left((\hat{\xi}^0+\rho^n)+\frac{\Delta t}{2}\sum_{k=0}^{n}(d_t\hat{\eta}^k-d_t\phi^k)+\frac{\Delta t}{2}\sum_{k=0}^{n-1}(d_t\hat{\eta}^k-d_t\phi^k)\right)\right),D_iV\right)\\
&+\frac{1}{8}\lambda^3(\Delta t)^6\left(q(\boldsymbol{x})\left(D_4\left((\hat{\xi}^0+\rho^n)+\frac{\Delta t}{2}\sum_{k=0}^{n}(d_t\hat{\eta}^k-d_t\phi^k)+\frac{\Delta t}{2}\sum_{k=0}^{n-1}(d_t\hat{\eta}^k-d_t\phi^k)\right)\right),D_4V\right)\\
&+\frac{\Delta t}{2}\left(\sum_{k=0}^{n}(a(u^k)-a(U^k))\nabla W^k+\sum_{k=0}^{n-1}(a(u^k)-a(U^k))\nabla W^k,\nabla V\right)\\
&=-F_1^n+(F_2^n,V)+\lambda\Delta t(F_3^n,V)+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(F_4^n,D_iV)+\frac{1}{8}\lambda^3(\Delta t)^6(F_5^n,D_4V)+\frac{\Delta t}{2}(F_6^n,\nabla V).
\end{aligned}
\tag{5.1}
$$

Let $V=\xi^{n+\frac{1}{2}}$, then (5.1) becomes:

$$
\begin{aligned}
&q_*\left(d_t\left((\xi^n,\xi^n)+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(D_i\xi^n,D_i\xi^n)+\frac{1}{8}\lambda^3(\Delta t)^6(D_4\xi^n,D_4\xi^n)\right)\right)\\
&\le -F_1^n+C\left[(F_2^n,V)+\lambda\Delta t(F_3^n,V)+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(F_4^n,D_iV)+\frac{1}{8}\lambda^3(\Delta t)^6(F_5^n,D_4V)+\frac{\Delta t}{2}(F_6^n,\nabla V)\right].
\end{aligned}
\tag{5.2}
$$

Multiplying by Δt, summing for $n=0,1\dots M-1$ $(0<M\le N)$ leads to

$$
\begin{aligned}
&\left\|\xi^M\right\|^2+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\xi^M\right\|^2+\frac{1}{8}\lambda^3(\Delta t)^6\left\|D_4\xi^M\right\|^2\\
&\le C\left(\left\|\xi^0\right\|^2+(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\xi^0\right\|^2+(\Delta t)^6\left\|D_4\xi^0\right\|^2-\Delta t\sum_{n=0}^{M-1}F_1^n\right)\\
&+C\Delta t\sum_{n=0}^{M-1}\left[(F_2^n,V)+\lambda\Delta t(F_3^n,V)+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(F_4^n,D_iV)+\frac{1}{8}\lambda^3(\Delta t)^6(F_5^n,D_4V)+\frac{\Delta t}{2}(F_6^n,\nabla V)\right],
\end{aligned}
\tag{5.3}
$$

F_1^n can be written as

$$
\begin{aligned}
F_1^n &= \lambda\Delta t\left(q(\boldsymbol{x})\nabla\left(\frac{\Delta t}{2}\sum_{k=0}^{n} d_t\xi^k + \frac{\Delta t}{2}\sum_{k=0}^{n-1} d_t\xi^k\right), \nabla V\right) + \frac{\Delta t}{2}\left(\left(\sum_{k=0}^{n} a(U^k)\nabla\xi^k + \sum_{k=0}^{n-1} a(U^k)\nabla\xi^k\right), \nabla V\right) \\
&= \frac{\Delta t}{2}\left(\sum_{k=0}^{n}(\lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^k, \nabla V) + (a(U^k)\nabla\xi^k, \nabla V)) + \sum_{k=0}^{n-1}(\lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^k, \nabla V) + (a(U^k)\nabla\xi^k, \nabla V))\right).
\end{aligned}
$$

Since $\xi^{k+\frac{1}{2}} = \xi^k + \frac{\Delta t}{2} d_t\xi^k$, hence

$$
\begin{aligned}
&\lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^k, \nabla V) + (a(U^k)\nabla\xi^k, \nabla V) = \lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^k, \nabla\xi^{k+\frac{1}{2}}) + (a(U^k)\nabla\xi^k, \nabla\xi^{k+\frac{1}{2}}) \\
&\quad = \lambda\Delta t\left(q(\boldsymbol{x})\nabla d_t\xi^k, \nabla\left(\xi^k + \frac{\Delta t}{2} d_t\xi^k\right)\right) + \left(a(U^k)\nabla\left(\xi^{k+\frac{1}{2}} - \frac{\Delta t}{2} d_t\xi^k\right), \nabla\xi^{k+\frac{1}{2}}\right) \\
&\quad = \lambda\Delta t(q(\boldsymbol{x})\nabla d_t\xi^k, \nabla\xi^k) + \lambda\frac{(\Delta t)^2}{2}(q(\boldsymbol{x})\nabla d_t\xi^k, \nabla d_t\xi^k) + (a(U^k)\nabla\xi^{k+\frac{1}{2}}, \nabla\xi^{k+\frac{1}{2}}) \\
&\qquad - \frac{(\Delta t)^2}{4}(a(U^k)\nabla d_t\xi^k, \nabla d_t\xi^k) - \frac{\Delta t}{2}(a(U^k)\nabla d_t\xi^k, \nabla\xi^k) \\
&\quad = \frac{(\Delta t)^2}{4}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla d_t\xi^k) + (a(U^k)\nabla\xi^{k+\frac{1}{2}}, \nabla\xi^{k+\frac{1}{2}}) + \frac{\Delta t}{2}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla\xi^k, \nabla d_t\xi^k),
\end{aligned}
$$

thus

$$
\begin{aligned}
-\Delta t\sum_{n=0}^{M-1} F_1^n = \sum_{n=0}^{M-1}&\left\{\sum_{k=0}^{n-1}\left\{-\frac{(\Delta t)^4}{8}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla d_t\xi^k) - \frac{(\Delta t)^2}{2}(a(U^k)\nabla\xi^{k+\frac{1}{2}}, \nabla\xi^{k+\frac{1}{2}})\right.\right. \\
&- \left.\frac{(\Delta t)^2}{4}(2\lambda q(\boldsymbol{x}) - a(U^k)\nabla\xi^k, \nabla d_t\xi^k)\right\} + \sum_{k=0}^{n}\left\{-\frac{(\Delta t)^4}{8}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla d_t\xi^k)\right. \\
&- \left.\left.\frac{(\Delta t)^2}{2}\left(a(U^k)\nabla\xi^{k+\frac{1}{2}}, \nabla\xi^{k+\frac{1}{2}}\right) - \frac{(\Delta t)^2}{4}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla\xi^k, \nabla d_t\xi^k)\right\}\right\}.
\end{aligned}
$$

From

$$
\begin{aligned}
\Delta t\sum_{k=0}^{n-1}(A^k, \nabla d_t\xi^k) &= (A^{n-1}, \nabla\xi^n) - (A^0, \nabla\xi^0) - \Delta t\sum_{k=0}^{n-2}(d_tA^k, \nabla\xi^k), \\
\Delta t\sum_{k=0}^{n}(A^k, \nabla d_t\xi^k) &= (A^n, \nabla\xi^{n+1}) - (A^0, \nabla\xi^0) - \Delta t\sum_{k=0}^{n-1}(d_tA^k, \nabla\xi^k),
\end{aligned}
$$

and denote that $F^k = (2\lambda q(\boldsymbol{x}) - a(U^k))\nabla\xi^k$, then

$$
\begin{aligned}
-\Delta t\sum_{n=0}^{M-1} F_1^n = \sum_{n=0}^{M-1}&\left\{\sum_{k=0}^{n-1}\left\{-\frac{(\Delta t)^4}{8}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla d_t\xi^k) - \frac{(\Delta t)^2}{2}(a(U^k)\nabla\xi^{k+\frac{1}{2}}, \nabla\xi^{k+\frac{1}{2}})\right\}\right. \\
&+ \left.\sum_{k=0}^{n}\left\{-\frac{(\Delta t)^4}{8}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla d_t\xi^k) - \frac{(\Delta t)^2}{2}(a(U^k)\nabla\xi^{k+\frac{1}{2}}, \nabla\xi^{k+\frac{1}{2}})\right\}\right\} \\
&- \frac{(\Delta t)^2}{4}\sum_{n=0}^{M-1}\{((2\lambda q(\boldsymbol{x}) - a(U^n))\nabla\xi^n, \nabla\xi^{n+1}) + ((2\lambda q(\boldsymbol{x}) - a(U^{n-1}))\nabla\xi^{n-1}, \nabla\xi^n) \\
&- 2((2\lambda q(\boldsymbol{x}) - a(U^0))\nabla\xi^0, \nabla\xi^0)\} + \frac{(\Delta t)^2}{4}\cdot\Delta t\sum_{n=0}^{M-1}\left\{\sum_{k=0}^{n}(d_tF^k, \nabla\xi^{k+1}) + \sum_{k=0}^{n-2}(d_tF^k, \nabla\xi^k)\right\},
\end{aligned}
$$

also

$$
\begin{aligned}
d_tF^k &= \frac{(2\lambda q(\boldsymbol{x}) - a(U^{k+1}))\nabla\xi^{k+1} - (2\lambda q(\boldsymbol{x}) - a(U^k))\nabla\xi^k}{\Delta t} \\
&= 2\lambda q(\boldsymbol{x})\nabla d_t\xi^k + \frac{a(U^k)\nabla\xi^k - a(U^{k+1})\nabla\xi^{k+1}}{\Delta t}
\end{aligned}
$$

$$= 2\lambda q(\boldsymbol{x})\nabla d_t\xi^k + \frac{a(U^k)\nabla\xi^k - a(U^k)\nabla\xi^{k+1} + a(U^k)\nabla\xi^{k+1} - a(U^{k+1})\nabla\xi^{k+1}}{\Delta t}$$
$$= (2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k - \frac{\partial a}{\partial u}\cdot d_tU^k\cdot\nabla\xi^{k+1},$$

thus

$$\begin{aligned}
-\Delta t\sum_{n=0}^{M-1}F_1^n &= \sum_{n=0}^{M-1}\left\{\sum_{k=0}^{n-1}\left\{-\frac{(\Delta t)^4}{8}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla d_t\xi^k) - \frac{(\Delta t)^2}{2}(a(U^k)\nabla\xi^{k+\frac{1}{2}}, \nabla\xi^{k+\frac{1}{2}})\right\}\right.\\
&\quad\left.+\sum_{k=0}^{n}\left\{-\frac{(\Delta t)^4}{8}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla d_t\xi^k) - \frac{(\Delta t)^2}{2}(a(U^k)\nabla\xi^{k+\frac{1}{2}}, \nabla\xi^{k+\frac{1}{2}})\right\}\right\}\\
&\quad-\frac{(\Delta t)^2}{4}\sum_{n=0}^{M-1}\{((2\lambda q(\boldsymbol{x}) - a(U^n))\nabla\xi^n, \nabla\xi^{n+1}) + ((2\lambda q(\boldsymbol{x}) - a(U^{n-1}))\nabla\xi^{n-1}, \nabla\xi^n)\\
&\quad-2((2\lambda q(\boldsymbol{x}) - a(U^0))\nabla\xi^0, \nabla\xi^0)\}\\
&\quad+\frac{(\Delta t)^2}{4}\cdot\Delta t\sum_{n=0}^{M-1}\left\{\sum_{k=0}^{n-1}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla\xi^{k+1}) + \sum_{k=0}^{n-2}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla\xi^k)\right.\\
&\quad\left.-\sum_{k=0}^{n-1}\left(\frac{\partial a}{\partial u}\cdot d_tU^k\cdot\nabla\xi^{k+1}, \nabla\xi^{k+1}\right) - \sum_{k=0}^{n-2}\left(\frac{\partial a}{\partial u}\cdot d_tU^k\cdot\nabla\xi^{k+1}, \nabla\xi^k\right)\right\}.
\end{aligned} \tag{5.4}$$

Substituting (5.4) into (5.3) yields

$$\begin{aligned}
&\left\|\xi^M\right\|^2 + \frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\xi^M\right\|^2 + \frac{1}{8}\lambda^3(\Delta t)^6\left\|D_4\xi^M\right\|^2\\
&\quad+\frac{(\Delta t)^4}{8}\sum_{n=0}^{M-1}\left\{\sum_{k=0}^{n-1}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla d_t\xi^k) + \sum_{k=0}^{n}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla d_t\xi^k)\right\}\\
&=\left(\left\|\xi^0\right\|^2 + (\Delta t)^4\sum_{i=1}^{3}\left\|D_i\xi^0\right\|^2 + (\Delta t)^6\left\|D_4\xi^0\right\|^2\right) - \frac{(\Delta t)^2}{2}\sum_{n=0}^{M-1}\left\{\sum_{k=0}^{n-1}\left(a(U^k)\nabla\xi^{k+\frac{1}{2}, \nabla\xi^{k+\frac{1}{2}}}\right)\right.\\
&\quad\left.+\sum_{k=0}^{n}(a(U^k)\nabla\xi^{k+\frac{1}{2}}, \nabla\xi^{k+\frac{1}{2}})\right\} - \frac{(\Delta t)^2}{4}\sum_{n=0}^{M-1}\{((2\lambda q(\boldsymbol{x}) - a(U^n))\nabla\xi^n, \nabla\xi^{n+1})\\
&\quad+((2\lambda q(\boldsymbol{x}) - a(U^{n-1}))\nabla\xi^{n-1}, \nabla\xi^n) - 2((2\lambda q(\boldsymbol{x}) - a(U^0))\nabla\xi^0, \nabla\xi^0)\}\\
&\quad+\frac{(\Delta t)^2}{4}\cdot\Delta t\sum_{n=0}^{M-1}\left\{\sum_{k=0}^{n-1}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla\xi^{k+1}) + \sum_{k=0}^{n-2}((2\lambda q(\boldsymbol{x}) - a(U^k))\nabla d_t\xi^k, \nabla\xi^k)\right.\\
&\quad\left.-\sum_{k=0}^{n-1}\left(\frac{\partial a}{\partial u}\cdot d_tU^k\cdot\nabla\xi^{k+1}, \nabla\xi^{k+1}\right) - \sum_{k=0}^{n-2}\left(\frac{\partial a}{\partial u}\cdot d_tU^k\cdot\nabla\xi^{k+1}, \nabla\xi^k\right)\right\}\\
&\quad+C\Delta t\sum_{n=0}^{M-1}\left[(F_2^n, V) + \lambda\Delta t(F_3^n, V) + \frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(F_4^n, D_iV) + \frac{1}{8}\lambda^3(\Delta t)^6(F_5^n, D_4V) + \frac{\Delta t}{2}(F_6^n, \nabla V)\right]\\
&\le\left(\left\|\xi^0\right\|^2 + (\Delta t)^4\sum_{i=1}^{3}\left\|D_i\xi^0\right\|^2 + (\Delta t)^6\left\|D_4\xi^0\right\|^2\right) + C(\Delta t)^2\sum_{n=0}^{M-1}\left\|\nabla\xi^{n+\frac{1}{2}}\right\|^2\\
&\quad+C(\Delta t)^2\left\|2\lambda q(\boldsymbol{x})I - A\right\|\sum_{n=0}^{M-1}\left(\left\|\nabla\xi^{n-1}\right\|^2 + \left\|\nabla\xi^n\right\|^2 + \left\|\nabla\xi^{n+1}\right\|^2\right) + C(\Delta t)^2\left\|\nabla\xi^0\right\|^2\\
&\quad+\varepsilon(\Delta t)^2\left\|2\lambda q(\boldsymbol{x})I - A\right\|\sum_{n=0}^{M-2}\left\|\nabla d_t\xi^n\right\|^2 + C(\Delta t)^2\sum_{n=0}^{M-2}(\left\|\nabla\xi^n\right\|^2 + \left\|\nabla\xi^{n+1}\right\|^2)\\
&\quad+C(\Delta t)^2\sum_{n=0}^{M-2}\left\|d_tU^n\right\|_{L^\infty}\left(\left\|\nabla\xi^n\right\|^2 + \left\|\nabla\xi^{n+1}\right\|^2\right)
\end{aligned}$$

$$+C\Delta t\sum_{n=0}^{M-1}\left[(F_2^n,V)+\lambda\Delta t(F_3^n,V)+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(F_4^n,D_iV)+\frac{1}{8}\lambda^3(\Delta t)^6(F_5^n,D_4V)+\frac{\Delta t}{2}(F_6^n,\nabla V)\right],$$

where I is a 3 by 3 identity matrix. A is a 3 by 3 matrix, which components are $a_{ij}(U^n)$.

$\lambda > \frac{1}{2}(\max_{x\in\Omega}\|A(\boldsymbol{x},U^n)\|/q_*)$, then

$$\begin{aligned}
&\left\|\xi^M\right\|^2+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\xi^M\right\|^2+\frac{1}{8}\lambda^3(\Delta t)^6\left\|D_4\xi^M\right\|^2+\frac{\hat{q}}{4}(\Delta t)^4\left\|\nabla d_t\xi^k\right\|^2\\
&\le\left(\left\|\xi^0\right\|^2+(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\xi^0\right\|^2+(\Delta t)^6\left\|D_4\xi^0\right\|^2\right)+C(\Delta t)^2\sum_{n=0}^{M-1}\left\|\nabla\xi^{n+\frac{1}{2}}\right\|^2\\
&\quad+C(\Delta t)^2\left\|2\lambda q(\boldsymbol{x})I-A\right\|\sum_{n=0}^{M-1}\left(\left\|\nabla\xi^{n-1}\right\|^2+\left\|\nabla\xi^n\right\|^2+\left\|\nabla\xi^{n+1}\right\|^2\right)+C(\Delta t)^2\left\|\nabla\xi^0\right\|^2\\
&\quad+\varepsilon(\Delta t)^2\left\|2\lambda q(\boldsymbol{x})I-A\right\|\sum_{n=0}^{M-2}\left\|\nabla d_t\xi^n\right\|^2+C(\Delta t)^2\sum_{n=0}^{M-2}\left(\left\|\nabla\xi^n\right\|^2+\left\|\nabla\xi^{n+1}\right\|^2\right)\\
&\quad+C(\Delta t)^2\sum_{n=0}^{M-2}\left\|d_tU^n\right\|_{L^\infty}\left(\left\|\nabla\xi^n\right\|^2+\left\|\nabla\xi^{n+1}\right\|^2\right)\\
&\quad+C\Delta t\sum_{n=0}^{M-1}\left[(F_2^n,V)+\lambda\Delta t(F_3^n,V)+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}(F_4^n,D_iV)+\frac{1}{8}\lambda^3(\Delta t)^6(F_5^n,D_4V)+\frac{\Delta t}{2}(F_6^n,\nabla V)\right].
\end{aligned}\tag{5.5}$$

Since

$$\begin{aligned}
\Delta t\sum_{n=0}^{M-1}\left(F_2^n,\xi^{n+\frac{1}{2}}\right)&\le C\Delta t\sum_{n=0}^{M-1}\left(\hat{\xi}^0+\rho^n+\frac{\Delta t}{2}\sum_{k=0}^{n}I^k+\frac{\Delta t}{2}\sum_{k=0}^{n-1}I^k,\xi^{n+\frac{1}{2}}\right)\\
&\le C\Delta t\sum_{n=0}^{M-1}\left\|\hat{\xi}^0\right\|^2+C\Delta t\sum_{n=0}^{M-1}\left\|\rho^n\right\|^2+C(\Delta t)^2\sum_{n=0}^{M-1}\left(\sum_{k=0}^{n}\left\|I^k\right\|^2+\sum_{k=0}^{n-1}\left\|I^k\right\|^2\right)+C\Delta t\sum_{n=0}^{M}\left\|\xi^n\right\|^2,
\end{aligned}\tag{5.6}$$

from (4.17),

$$\Delta t\sum_{n=0}^{M-1}\left\|\rho^n\right\|^2\le C[(\Delta t)^4+h^{2r}],\tag{5.7}$$

from (4.5) and (4.12), assuming that $\Delta t=O(h^2)$, then

$$\begin{aligned}
(\Delta t)^2\sum_{n=0}^{M-1}\sum_{k=0}^{n}\left\|I^k\right\|^2&\le C(\Delta t)^2\left(\sum_{n=0}^{M-1}\left\|\left(\frac{\partial\phi}{\partial t}\right)^n-d_t\phi^n+d_t\hat{\eta}^n\right\|^2+\sum_{n=0}^{M-1}\left\|\xi^n\right\|_1^2+\sum_{n=0}^{M-1}\left\|\eta^n\right\|^2\right)\\
&\le C[(\Delta t)^2+h^{2r}]+C(\Delta t)^2h^{-2}\sum_{n=0}^{M-1}\left\|\xi^n\right\|^2+C(\Delta t)^2\\
&\le C[(\Delta t)^2+h^{2r}]+C\Delta t\sum_{n=0}^{M-1}\left\|\xi^n\right\|^2,
\end{aligned}\tag{5.8}$$

from (5.6)–(5.8), it follows that

$$\Delta t\sum_{n=0}^{M-1}\left(F_2^n,\xi^{n+\frac{1}{2}}\right)\le C[(\Delta t)^2+h^{2r}]+C\Delta t\sum_{n=0}^{M}\left\|\xi^n\right\|^2\tag{5.9}$$

provided that $\left\|\hat{\xi}^0\right\|=O(h^r)$. Also

$$\begin{aligned}
\Delta t\sum_{n=0}^{M-1}\lambda\Delta t(F_3^n,V)&\le C(\Delta t)^2\left[\sum_{n=0}^{M-1}\left\|F_3^n\right\|^2+\sum_{n=0}^{M}\left\|\xi^n\right\|^2\right]\\
&=C(\Delta t)^2\left[\sum_{n=0}^{M-1}\left\|\Delta(u^{n+\frac{1}{2}}-u^0-\eta^{n+\frac{1}{2}}+\eta^0)\right\|^2+\sum_{n=0}^{M}\left\|\xi^n\right\|^2\right]
\end{aligned}$$

$$\leq C(\Delta t)^2(\|u\|^2_{L^2(H^r)} + \|u\|^2_{L^\infty(H^r)} + \|\eta\|^2_{L^2(H^r)} + \|\eta\|^2_{L^\infty(H^r)}) + C(\Delta t)^2 \sum_{n=0}^{M} \left\|\xi^n\right\|^2$$

$$\leq C(\Delta t)^2 + C\Delta t \sum_{n=0}^{M} \left\|\xi^n\right\|^2. \tag{5.10}$$

Assume that $(\Delta t)^2 \sum_{i=1}^{3} \left\|D_i\hat{\xi}^0\right\| = O(h^r)$, then

$$\frac{1}{4}\lambda^2 \Delta t \sum_{n=0}^{M-1} \left((\Delta t)^4 \sum_{i=1}^{3} (F_4^n, D_i V)\right) \leq C\Delta t \sum_{n=0}^{M-1} (\Delta t)^4 \left[\sum_{i=1}^{3} \left\|F_4^n\right\|^2 + \sum_{i=1}^{3} \left\|D_i \xi^{n+\frac{1}{2}}\right\|^2\right]$$

$$\leq C\Delta t \sum_{n=0}^{M-1} \left[(\Delta t)^4 \sum_{i=1}^{3} \left\|F_4^n\right\|^2\right] + C\Delta t \sum_{n=0}^{M} \left[(\Delta t)^4 \sum_{i=1}^{3} \left\|D_i \xi^n\right\|^2\right]$$

$$= C\Delta t \sum_{n=0}^{M-1} \left\{(\Delta t)^4 \sum_{i=1}^{3} \left\|D_i \left[(\hat{\xi}^0 + \rho^n) + \frac{\Delta t}{2} \sum_{k=0}^{n} (d_t \hat{\eta}^k - d_t \phi^k) + \frac{\Delta t}{2} \sum_{k=0}^{n-1} (d_t \hat{\eta}^k - d_t \phi^k)\right]\right\|^2\right\}$$

$$+ C\Delta t \sum_{n=0}^{M} \left[(\Delta t)^4 \sum_{i=1}^{3} \left\|D_i \xi^n\right\|^2\right]$$

$$= C\Delta t \sum_{n=0}^{M-1} \left[(\Delta t)^4 \sum_{i=1}^{3} \left\|D_i[(\hat{\xi}^0 + \rho^n) + \hat{\eta}^{n+\frac{1}{2}} - \hat{\eta}^0 + \phi^{n+\frac{1}{2}} + \phi^0]\right\|^2\right] + C\Delta t \sum_{n=0}^{M} \left[(\Delta t)^4 \sum_{i=1}^{3} \left\|D_i \xi^n\right\|^2\right]$$

$$\leq C(\Delta t)^4 \sum_{i=1}^{3} \left[(\Delta t)^2 + \left\|D_i \hat{\xi}^0\right\|^2 + \left\|D_i \left(\frac{\partial \eta}{\partial t}\right)\right\|^2_{L^2(L^2)} + \left\|D_i \left(\frac{\partial \eta}{\partial t}\right)\right\|^2_{L^\infty(L^2)}\right.$$

$$\left. + \left\|D_i \left(\frac{\partial u}{\partial t}\right)\right\|^2_{L^2(L^2)} + \left\|D_i \left(\frac{\partial u}{\partial t}\right)\right\|^2_{L^\infty(L^2)}\right] + C\Delta t \sum_{n=0}^{M} \left[(\Delta t)^4 \sum_{i=1}^{3} \left\|D_i \xi^n\right\|^2\right]$$

$$\leq C[(\Delta t)^2 + h^{2r}] + C\Delta t \sum_{n=0}^{M} \left[(\Delta t)^4 \sum_{i=1}^{3} \left\|D_i \xi^n\right\|^2\right]. \tag{5.11}$$

Similarly, assume that $(\Delta t)^3 \left\|D_4\hat{\xi}^0\right\| = O(h^r)$, then for $r \geq 3$ we have

$$\frac{1}{8}\lambda^3 \Delta t \sum_{n=0}^{M-1} ((\Delta t)^6 (F_5^n, D_4 V)) \leq C\Delta t \sum_{n=0}^{M-1} (\Delta t)^6 \left[\left\|F_5^n\right\|^2 + \left\|D_4 \xi^{n+\frac{1}{2}}\right\|^2\right]$$

$$\leq C\Delta t \sum_{n=0}^{M-1} \left[(\Delta t)^6 \left\|F_5^n\right\|^2\right] + C\Delta t \sum_{n=0}^{M} \left[(\Delta t)^6 \left\|D_4 \xi^n\right\|^2\right]$$

$$= C\Delta t \sum_{n=0}^{M-1} \left\{(\Delta t)^6 \left\|D_4 \left[(\hat{\xi}^0 + \rho^n) + \frac{\Delta t}{2} \sum_{k=0}^{n} (d_t \hat{\eta}^k - d_t \phi^k) + \frac{\Delta t}{2} \sum_{k=0}^{n-1} (d_t \hat{\eta}^k - d_t \phi^k)\right]\right\|^2\right\}$$

$$+ C\Delta t \sum_{n=0}^{M} \left[(\Delta t)^6 \left\|D_4 \xi^n\right\|^2\right]$$

$$= C\Delta t \sum_{n=0}^{M-1} \left[(\Delta t)^6 \left\|D_4[(\hat{\xi}^0 + \rho^n) + \hat{\eta}^{n+\frac{1}{2}} - \hat{\eta}^0 + \phi^{n+\frac{1}{2}} + \phi^0]\right\|^2\right] + C\Delta t \sum_{n=0}^{M} \left[(\Delta t)^6 \left\|D_4 \xi^n\right\|^2\right]$$

$$\leq C(\Delta t)^6 \left[(\Delta t)^2 + \left\|D_4 \hat{\xi}^0\right\|^2 + \left\|D_4 \left(\frac{\partial \eta}{\partial t}\right)\right\|^2_{L^2(L^2)} + \left\|D_4 \left(\frac{\partial \eta}{\partial t}\right)\right\|^2_{L^\infty(L^2)}\right.$$

$$\left. + \left\|D_4 \left(\frac{\partial u}{\partial t}\right)\right\|^2_{L^2(L^2)} + \left\|D_4 \left(\frac{\partial u}{\partial t}\right)\right\|^2_{L^\infty(L^2)}\right] + C\Delta t \sum_{n=0}^{M} \left[(\Delta t)^6 \left\|D_4 \xi^n\right\|^2\right]$$

$$\leq C[(\Delta t)^2 + h^{2r}] + C\Delta t \sum_{n=0}^{M} \left[(\Delta t)^6 \left\|D_4 \xi^n\right\|^2\right]. \tag{5.12}$$

When $\Delta t = O(h^2)$,

$$C\Delta t\cdot\frac{\Delta t}{2}\sum_{n=0}^{M-1}(F_6^n,\nabla V)\le C(\Delta t)^2\left(\sum_{n=0}^{M-1}\left\|F_6^n\right\|^2+\sum_{n=0}^{M}\left\|\xi^n\right\|_1^2\right)$$
$$\le C(\Delta t)^2\left(\sum_{n=0}^{M-1}\left\|\eta^n\right\|_1^2+\sum_{n=0}^{M}\left\|\xi^n\right\|_1^2\right)\le C(\Delta t)^2h^{-2}\left(\sum_{n=0}^{M-1}\left\|\eta^n\right\|^2+\sum_{n=0}^{M}\left\|\xi^n\right\|^2\right)$$
$$\le Ch^{2r}+C\Delta t\sum_{n=0}^{M}\left\|\xi^n\right\|^2. \tag{5.13}$$

From (5.5), and combining (5.9)–(5.13) yields:

$$\left\|\xi^M\right\|^2+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\xi^M\right\|^2+\frac{1}{8}\lambda^3(\Delta t)^6\left\|D_4\xi^M\right\|^2+C(\Delta t)^2\sum_{n=0}^{M-1}\left\|d_t\xi^n\right\|_{H_0^1}^2$$
$$\le C\left(\left\|\xi^0\right\|^2+(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\xi^0\right\|^2+(\Delta t)^6\left\|D_4\xi^0\right\|^2\right)+C[(\Delta t)^2+h^{2r}]+C\Delta t\sum_{n=0}^{M}\left\|\xi^n\right\|^2$$
$$+C\Delta t\sum_{n=0}^{M-2}\left\|d_tU^n\right\|_{L^\infty}\left(\left\|\xi^n\right\|^2+\left\|\xi^{n+1}\right\|^2\right)+C\Delta t\sum_{n=0}^{M}\left[(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\xi^n\right\|^2+(\Delta t)^6\left\|D_4\xi^n\right\|^2\right]. \tag{5.14}$$

Introduce the induction hypothesis:

$$\sup_{1\le s\le M-1}\Delta t\sum_{n=1}^{s}\left\|d_tU^{n-1}\right\|_{L^\infty}\le C. \tag{5.15}$$

If $\left\|\xi^0\right\|$, $\left\|\hat{\xi}^0\right\|$, $(\Delta t)^2\sum_{i=1}^3\left\|D_i\xi^0\right\|$, $(\Delta t)^3\left\|D_4\xi^0\right\|$, $(\Delta t)^2\sum_{i=1}^3\left\|D_i\hat{\xi}^0\right\|$ and $(\Delta t)^3\left\|D_4\hat{\xi}^0\right\|$ are $O(h^r)$, and using Gronwall inequality, we have

$$\left\|\xi^M\right\|^2+\frac{1}{4}\lambda^2(\Delta t)^4\sum_{i=1}^{3}\left\|D_i\xi^M\right\|^2+\frac{1}{8}\lambda^3(\Delta t)^6\left\|D_4\xi^M\right\|^2+(\Delta t)^2\sum_{n=0}^{M-1}\left\|d_t\xi^n\right\|_{H_0^1}^2\le C[(\Delta t)^2+h^{2r}]. \tag{5.16}$$

Then from Lemma 1 and by triangle inequality, it leads to

$$\max_{0\le n\le N}\left\|U^n-u^n\right\|\le\max_{0\le n\le N}\left[\left\|\xi^n\right\|+\left\|\eta^n\right\|\right]\le C[\Delta t+h^r].$$

Similar to Theorem 1, we can also prove the induction hypothesis

$$\max_{0\le n\le M}\|U^n-u^n\|_{L^\infty}\le\varepsilon_0\quad\text{and}\quad\sup_{1\le s\le M-1}\Delta t\sum_{n=1}^{s}\|d_tU^{n-1}\|_{L^\infty}\le C.\quad\square$$

Thus, the following theorem is obtained:

Theorem 2. *Assume that u is the solution to (1.1), let U and W be defined by (3.5) and (2.2), respectively, if $u\in C^4(\bar{\Omega}\times[0,T])$, u, $\frac{\partial u}{\partial t}\in L^\infty(H^r)$, $\frac{\partial^2 u}{\partial t^2}\in L^2(H^r)(r\ge 3)$, $\Delta t=O(h^2)$ and $\lambda>\frac{1}{2}(\max_{x\in\Omega}\|A(\boldsymbol{x},U^n)\|/q_*)$, if h is sufficiently small, and*

$$\left\|U^0-W^0\right\|+\left\|\Phi^0-\left(\frac{\partial W}{\partial t}\right)^0\right\|+(\Delta t)^2\sum_{i=1}^{3}\left\|D_i[U^0-W^0]\right\|+(\Delta t)^3\left\|D_4[U^0-W^0]\right\|$$
$$+(\Delta t)^2\sum_{i=1}^{3}\left\|D_i\left[\Phi^0-\left(\frac{\partial W}{\partial t}\right)^0\right]\right\|+(\Delta t)^3\left\|D_4\left[\Phi^0-\left(\frac{\partial W}{\partial t}\right)^0\right]\right\|\le C[\Delta t+h^r],$$

there exists a positive constant C such that

$$\max_{0\le n\le N}\left\|U^n-u^n\right\|\le C[\Delta t+h^r].$$

6. Matrix problems

Assume that $S_{h,r} = S^x_{h,r} \otimes S^y_{h,r} \otimes S^z_{h,r}$, where $S^x_{h,r}, S^y_{h,r}$ and $S^z_{h,r}$ are the finite-dimensional subspaces of $H^1_0([a,b]), H^1_0([c,d]), H^1_0([e,f])$, respectively. Let $\left\{\gamma^1_p(x)\gamma^2_q(y)\gamma^3_t(z)\right\}^{N_x,N_y,N_z}_{p=1,q=1.t=1}$ be the tensor product basis, where $\left\{\gamma^1_p(x)\right\}^{N_x}_{p=1}$, $\left\{\gamma^2_q(y)\right\}^{N_y}_{q=1}$ and $\left\{\gamma^3_t(z)\right\}^{N_z}_{t=1}$ are the basis of $S^x_{h,r}, S^y_{h,r}$ and $S^z_{h,r}$, respectively. p, q and t are the grid line numbers in x-direction, y-direction and z-direction, respectively, $p = 1, 2, \ldots, N_x$, $q = 1, 2, \ldots, N_y$, $t = 1, 2, \ldots, N_z$.

Let

$$U^n(x,y,z) = \sum_{p,q,t} \alpha^{(n)}_{pqt}\gamma^1_p(x)\gamma^2_q(y)\gamma^3_t(z), \qquad \Phi^n(x,y,z) = \sum_{p,q,t} \beta^{(n)}_{pqt}\gamma_p^1(x)(y)\gamma^3_t(z). \tag{6.1}$$

Then

$$E^{n+1}(x,y,z) = \sum_{p,q,t} \theta^{(n+1)}_{pqt}\gamma^1_p(x)\gamma^2_q(y)\gamma^3_t(z), \quad \theta^{(n+1)}_{pqt} = \beta^{(n+1)}_{pqt} - \beta^{(n)}_{pqt}. \tag{6.2}$$

Choosing $V = \gamma^1_k(x)\gamma^2_m(y)\gamma^3_n(z)$, $k = 1, 2, \ldots, N_x$, $m = 1, 2, \ldots, N_y$, $n = 1, 2, \ldots, N_z$, then (3.5) becomes

$$\begin{aligned}
&\sum_{p=1}^{N_x}\sum_{q=1}^{N_y}\sum_{t=1}^{N_z}\Bigg\{(q(\boldsymbol{x})\gamma^1_p(x)\gamma^2_q(y)\gamma^3_t(z), \gamma^1_k(x)\gamma^2_m(y)\gamma^3_n(z)) \\
&\quad + \frac{\lambda}{2}(\Delta t)^2(q(\boldsymbol{x})\nabla(\gamma^1_p(x)\gamma^2_q(y)\gamma^3_t(z)), \nabla(\gamma^1_k(x)\gamma^2_m(y)\gamma^3_n(z))) \\
&\quad + \frac{\lambda^2}{4}(\Delta t)^4\sum_{i=1}^{3}(q(\boldsymbol{x})D_i(\gamma^1_p(x)\gamma^2_q(y)\gamma^3_t(z)), D_i(\gamma^1_k(x)\gamma^2_m(y)\gamma^3_n(z))) \\
&\quad + \frac{1}{8}\lambda^3(\Delta t)^6(q(\boldsymbol{x})D_4(\gamma^1_p(x)\gamma^2_q(y)\gamma^3_t(z)), D_4(\gamma^1_k(x)\gamma^2_m(y)\gamma^3_n(z)))\Bigg\}\theta^{(n+1)}_{pqt} = \Psi^{(n)}_{pqt},
\end{aligned} \tag{6.3a}$$

$$\sum_{p=1}^{N_x}\sum_{q=1}^{N_y}\sum_{t=1}^{N_z}\gamma^1_p(x)\gamma^2_q(y)\gamma^3_t(z)\alpha^{(n+1)}_{pqt} = \sum_{p=1}^{N_x}\sum_{q=1}^{N_y}\sum_{t=1}^{N_z}\gamma^1_p(x)\gamma^2_q(y)\gamma^3_t(z)\left[\alpha^{(n)}_{pqt} + \Delta t\left(\beta^{(n)}_{pqt} + \frac{1}{2}\theta^{(n+1)}_{pqt}\right)\right], \tag{6.3b}$$

where

$$\Psi^{(n)}_{pqt} = \Delta t\left[(f(U^n), V) - \sum_{i,j=1}^{3}\left(a_{ij}(U^n)\frac{\partial U^n}{\partial x_j}, \frac{\partial V}{\partial x_i}\right) - \sum_{i=1}^{3}\left(b_i(U^n)\frac{\partial U^n}{\partial x_i}, V\right) - \lambda\Delta t(q(\boldsymbol{x})\nabla\Phi^n, \nabla V)\right]. \tag{6.3c}$$

So that the matrix form of (6.3) is:

$$D^{\frac{1}{2}}\left[\left(C_1 + \frac{1}{2}\lambda(\Delta t)^2A_1\right)\otimes\left(C_2 + \frac{1}{2}\lambda(\Delta t)^2A_2\right)\otimes\left(C_3 + \frac{1}{2}\lambda(\Delta t)^2A_3\right)\right]D^{\frac{1}{2}}\theta^{(n+1)} = \Psi^{(n)}_{pqt}, \tag{6.4a}$$

$$\alpha^{(n+1)} = \alpha^{(n)} + \Delta t\left[\beta^{(n)} + \frac{1}{2}\theta^{(n+1)}\right], \quad n = 0, 1, 2, \ldots \tag{6.4b}$$

where

$$\begin{aligned}
C_1 &= \left\{\int_a^b \gamma^1_p(x)\gamma^1_q(x)\mathrm{d}x\right\}, \qquad A_1 = \left\{\int_a^b (\gamma^1_p(x))'_x(\gamma^1_q(x))'_x\mathrm{d}x\right\}, \\
C_2 &= \left\{\int_c^d \gamma^2_p(y)\gamma^2_q(y)\mathrm{d}y\right\}, \qquad A_2 = \left\{\int_c^d (\gamma^2_p(y))'_y(\gamma^2_q(y))'_y\mathrm{d}y\right\}, \\
C_3 &= \left\{\int_e^f \gamma^3_p(z)\gamma^3_q(z)\mathrm{d}z\right\}, \qquad A_3 = \left\{\int_e^f (\gamma^3_p(z))'_z(\gamma^3_q(z))'_z\mathrm{d}z\right\}, \\
D &= \begin{bmatrix} q(\boldsymbol{x}^1) & & & 0 \\ & q(\boldsymbol{x}^2) & & \\ & & \ddots & \\ 0 & & & q(\boldsymbol{x}^{N_xN_yN_z}) \end{bmatrix}
\end{aligned}$$

(6.4a) can be decomposed as:

$$D^{\frac{1}{2}}\left[I_{N_x}\otimes I_{N_y}\otimes\left(C_3+\frac{1}{2}\lambda(\Delta t)^2A_3\right)\right]\left[I_{N_x}\otimes\left(C_2+\frac{1}{2}\lambda(\Delta t)^2A_2\right)\otimes I_{N_z}\right]$$
$$\times\left[\left(C_1+\frac{1}{2}\lambda(\Delta t)^2A_1\right)\otimes I_{N_y}\otimes I_{N_z}\right]D^{\frac{1}{2}}\theta^{(n+1)}=\Psi_{pqt}^{(n)}, \tag{6.5}$$

where I_{N_x} is an identity matrix of order N_x, I_{N_y} is an identity matrix of order N_y, I_{N_z} is an identity matrix of order N_z. Then (6.5) can be solved by

$$D^{\frac{1}{2}}K_3K_2K_1D^{\frac{1}{2}}\theta^{(n+1)}=\Psi_{pqt}^{(n)},\quad n=0,1,2,\ldots. \tag{6.6}$$

If the nodes in Ω are firstly numbered in vertical order and the linear tensor product basis is

$$\gamma_p^1(x)=\begin{cases}0, & x\in[a,x_{p-1}]\\ (x-x_{p-1})/h, & x\in(x_{p-1},x_p]\\ (x_{p+1}-x)/h, & x\in(x_p,x_{p+1}]\\ 0, & x\in(x_{p+1},b]\end{cases}\qquad \gamma_q^2(y)=\begin{cases}0, & y\in[c,y_{q-1}]\\ (y-y_{q-1})/h, & y\in(y_{q-1},y_q]\\ (y_{q+1}-y)/h, & y\in(y_q,y_{q+1}]\\ 0, & y\in(y_{q+1},d]\end{cases}$$
$$\gamma_t^3(z)=\begin{cases}0, & z\in[e,z_{t-1}]\\ (z-z_{t-1})/h, & z\in(z_{t-1},z_t]\\ (z_{t+1}-z)/h, & z\in(z_t,z_{t+1}]\\ 0, & z\in(z_{t+1},f]\end{cases} \tag{6.7}$$

then the matrices in (6.6) are

$$K_1=\begin{bmatrix}X_{11} & X_{12} & & \cdots & 0\\ X_{21} & X_{22} & & & \\ & & \ddots & & \\ \vdots & & & X_{N_x-1N_x-1} & X_{N_x-1N_x}\\ & 0 & & X_{N_xN_x-1} & X_{N_xN_x}\end{bmatrix},\qquad K_2=\begin{bmatrix}Y_1 & & \cdots & 0\\ & Y_2 & & \\ \vdots & & \ddots & \\ 0 & & & Y_{N_x}\end{bmatrix}$$
$$K_3=\begin{bmatrix}Z_1 & & \cdots & 0\\ & Z_2 & & \\ \vdots & & \ddots & \\ 0 & & & Z_{N_x}\end{bmatrix}$$

where the components of K_1, K_2, K_3 are:

$$X_{st}=\begin{bmatrix}x_{st} & & & 0\\ & x_{st} & & \\ & & \ddots & \\ 0 & & & x_{st}\end{bmatrix}_{N_yN_z\times N_yN_z},\qquad Y_l=\begin{bmatrix}\begin{bmatrix}y_{11} & & 0\\ & \ddots & \\ 0 & & y_{11}\end{bmatrix}_{N_z\times N_z} & \cdots & \begin{bmatrix}y_{1N_y} & & 0\\ & \ddots & \\ 0 & & y_{1N_y}\end{bmatrix}_{N_z\times N_z}\\ \vdots & \ddots & \vdots\\ \begin{bmatrix}y_{N_y1} & & 0\\ & \ddots & \\ 0 & & y_{N_y1}\end{bmatrix}_{N_z\times N_z} & \cdots & \begin{bmatrix}y_{N_yN_y} & & 0\\ & \ddots & \\ 0 & & y_{N_yN_y}\end{bmatrix}_{N_z\times N_z}\end{bmatrix},$$

$$Z_l=\begin{bmatrix}\begin{bmatrix}z_{11} & \cdots & z_{1N_z}\\ \vdots & \ddots & \vdots\\ z_{N_z1} & \cdots & z_{N_zN_z}\end{bmatrix} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \begin{bmatrix}z_{11} & \cdots & z_{1N_z}\\ \vdots & \ddots & \vdots\\ z_{N_z1} & \cdots & z_{N_zN_z}\end{bmatrix}\end{bmatrix}_{N_yN_z\times N_yN_z}$$

$$\boldsymbol{x}_{st}=\int_a^b\left[\gamma_s^1(x)\gamma_t^1(x)+\frac{1}{2}\lambda(\Delta t)^2(\gamma_s^1(x))'_x(\gamma_t^1(x))'_x\right]\mathrm{d}x,$$
$$y_{kn}=\int_c^d\left[\gamma_k^2(y)\gamma_n^2(y)+\frac{1}{2}\lambda(\Delta t)^2(\gamma_k^2(y))'_y(\gamma_n^2(y))'_y\right]\mathrm{d}y,$$
$$z_{kn}=\int_e^f\left[\gamma_k^3(z)\gamma_n^3(z)+\frac{1}{2}\lambda(\Delta t)^2(\gamma_k^3(z))'_z(\gamma_n^3(z))'_z\right]\mathrm{d}z.$$

7. Numerical example

Consider

$$\begin{cases} (x^2+y^2+z^2+1)u_{tt} - \nabla\cdot((u+1)\nabla u) = f(u,x,y,z,t), & \Omega: 0 \le x,y,z \le \dfrac{1}{2}, t\in(0,1] \\ u(x,y,z,t) = 0, & (x,y,z)\in\partial\Omega \\ u(x,y,z,0) = \sin(2\pi x)\cdot\sin(2\pi y)\cdot\sin(2\pi z), & \\ u_t(x,y,z,0) = \sin(2\pi x)\cdot\sin(2\pi y)\cdot\sin(2\pi z) & \end{cases} \tag{7.1}$$

where

$$\begin{aligned} f(u,\boldsymbol{x},y,z,t) = {} & (x^2+y^2+z^2+1)u - \pi^2 e^{2t}[\cos(4\pi x)+\cos(4\pi y)+\cos(4\pi z)] + 2\pi^2 e^{2t}[\cos(4\pi x)\cos(4\pi y) \\ & + \cos(4\pi x)\cos(4\pi z) + \cos(4\pi y)\cos(4\pi z)] - 3\pi^2 e^{2t}\cos(4\pi x)\cos(4\pi y)\cos(4\pi z), \end{aligned}$$

the exact solution of (7.1) is:

$$u = e^t \sin(2\pi x)\sin(2\pi y)\sin(2\pi z).$$

And the column vector in right-hand side of (6.3c) is ordered according to the nodes: $(1,1,1),\ldots,(1,1,N_z)$, $(2,1,1),\ldots,(N_x,1,1),\ldots,(N_x,N_y,N_z)$, thus for $1\le k\le N_x$, $1\le m\le N_y$ and $1\le n\le N_z$, the terms in right-hand side of (6.3c) can be written as

$$(f(U^n),V) = (f,\gamma_k^1(x)\gamma_m^2(y)\gamma_n^3(z)) = \int_a^b\int_c^d\int_e^f f\cdot\gamma_k^1(x)\gamma_m^2(y)\gamma_n^3(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{7.2}$$

$$\begin{aligned} (U^n\nabla U^n,\nabla V) &= \left(U^n\frac{\partial U^n}{\partial x},\frac{\partial V}{\partial x}\right) + \left(U^n\frac{\partial U^n}{\partial y},\frac{\partial V}{\partial y}\right) + \left(U^n\frac{\partial U^n}{\partial z},\frac{\partial V}{\partial z}\right) \\ &= \sum_{p,q,t}\Bigg[(\alpha_{pqt}^{(n)})^2\int_a^b\gamma_p^1(x)(\gamma_p^1(x))'_x(\gamma_k^1(x))'_x\mathrm{d}x\int_c^d\gamma_q^2(y)\gamma_q^2(y)\gamma_m^2(y)\mathrm{d}y\int_e^f\gamma_t^2(z)\gamma_t^2(z)\gamma_n^3(z)\mathrm{d}z \\ &\quad + (\alpha_{pqt}^{(n)})^2\int_a^b\gamma_p^1(x)\gamma_p^1(x)\gamma_k^1(x)\mathrm{d}x\int_c^d\gamma_q^2(y)(\gamma_q^2(y))'_y(\gamma_m^2(y))'_y\mathrm{d}y\int_e^f\gamma_t^2(z)\gamma_t^2(z)\gamma_n^3(z)\mathrm{d}z \\ &\quad + (\alpha_{pqt}^{(n)})^2\int_a^b\gamma_p^1(x)\gamma_p^1(x)\gamma_k^1(x)\mathrm{d}x\int_c^d\gamma_q^2(y)\gamma_q^2(y)\gamma_m^2(y)\mathrm{d}y\int_e^f\gamma_t^2(z)(\gamma_t^2(z))'_z(\gamma_n^3(z))'_z\mathrm{d}z\Bigg], \end{aligned} \tag{7.3}$$

where

$$\int_a^b\gamma_p^1(x)(\gamma_p^1(x))'_x(\gamma_k^1(x))'_x\mathrm{d}x = \begin{cases} 0, & p\le k-2 \\ -(x_k^2-x_{k-1}^2-2x_kh)/(2h^3), & p=k-1 \\ (4h^2+2x_k^2-x_{k-1}^2-x_{k+1}^2)/(2h^3), & p=k \\ -(x_{k+1}^2-x_k^2-2x_kh)/(2h^3), & p=k+1 \\ 0, & p\ge k+2 \end{cases}$$

$$\begin{aligned} &\int_c^d\gamma_q^2(y)\gamma_q^2(y)\gamma_m^2(y)\mathrm{d}y \\ &= \begin{cases} 0, \quad q\le m-2 \\ \left(\dfrac{1}{12}y_m^4+\dfrac{1}{12}y_{m-1}^4-\dfrac{1}{3}y_{m-1}y_m^3-\dfrac{1}{3}y_my_{m-1}^3+\dfrac{1}{2}y_{m-1}^2y_m^2\right)/h^3, \quad q=m-1 \\ \dfrac{1}{h^3}\left(\dfrac{1}{2}y_m^4+\dfrac{1}{4}y_{m-1}^4+\dfrac{1}{4}y_{m+1}^4-y_{m-1}y_m^3-y_my_{m+1}^3+\dfrac{3}{2}y_{m-1}^2y_m^2+\dfrac{3}{2}y_m^2y_{m+1}^2-y_my_{m-1}^3-y_{m+1}y_m^3\right), \quad q=m \\ \left(\dfrac{1}{12}y_m^4+\dfrac{1}{12}y_{m+1}^4-\dfrac{1}{3}y_{m+1}y_m^3-\dfrac{1}{3}y_my_{m+1}^3+\dfrac{1}{2}y_m^2y_{m+1}^2\right)/h^3, \quad q=m+1 \\ 0, \quad q\ge m+2 \end{cases} \end{aligned}$$

$\int_a^b\gamma_p^1(x)\gamma_p^1(x)\gamma_k^1(x)\mathrm{d}x$, $\int_e^f\gamma_t^2(z)\gamma_t^2(z)\gamma_n^3(z)\mathrm{d}z$, $\int_c^d\gamma_q^2(y)(\gamma_q^2(y))'_y(\gamma_m^2(y))'_y\mathrm{d}y$ and $\int_e^f\gamma_t^2(z)(\gamma_t^2(z))'_z(\gamma_n^3(z))'_z\mathrm{d}z$ can be written in the similar way. Also

$$\begin{aligned} &\lambda\Delta t((x^2+y^2+z^2+1)\nabla\Phi^n,\nabla V) = \lambda\Delta t((\boldsymbol{x}^2\nabla\Phi^n,\nabla V)+(y^2\nabla\Phi^n,\nabla V)+(z^2\nabla\Phi^n,\nabla V)+(\nabla\Phi^n,\nabla V)) \\ &= \lambda\Delta t\sum_{p,q,t}\beta_{pqt}^{(n)}\int_a^b\int_c^d\int_e^f x^2[(\gamma_p^1(x))'_x\gamma_q^2(y)\gamma_t^3(z)(\gamma_k^1(x))'_x\gamma_m^2(y)\gamma_n^3(z) \\ &\quad + \gamma_p^1(x)(\gamma_q^2(y))'_y\gamma_t^3(z)\gamma_k^1(x)(\gamma_m^2(y))'_y\gamma_n^3(z) + \gamma_p^1(x)\gamma_q^2(y)(\gamma_t^3(z))'_z\gamma_k^1(x)\gamma_m^2(y)(\gamma_n^3(z))'_z]\mathrm{d}x\mathrm{d}y\mathrm{d}z \end{aligned}$$

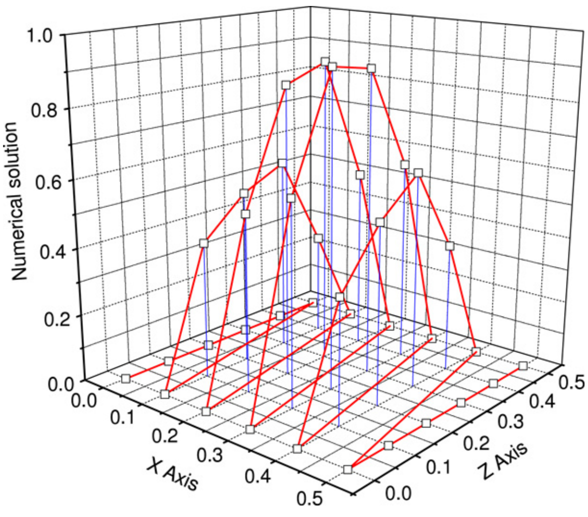

Fig. 1. The numerical solutions at $y = 0.3$, $t = 0.05$.

Table 1
The maximum absolute error, the average absolute error and the L^2 norm error at $y = 0.3$, $t = 0.05$.

t	Δt	Maximum absolute error	Average absolute error	L^2 norm error
0.05	0.000001	0.0735	0.00823059	0.0277484

$$
\begin{aligned}
&+\lambda\Delta t\sum_{p,q,t}\beta_{pqt}^{(n)}\int_a^b\int_c^d\int_e^f y^2[(\gamma_p^1(x))'_x\gamma_q^2(y)\gamma_t^3(z)(\gamma_k^1(x))'_x\gamma_m^2(y)\gamma_n^3(z)\\
&+\gamma_p^1(x)(\gamma_q^2(y))'_y\gamma_t^3(z)\gamma_k^1(x)(\gamma_m^2(y))'_y\gamma_n^3(z)+\gamma_p^1(x)\gamma_q^2(y)(\gamma_t^3(z))'_z\gamma_k^1(x)\gamma_m^2(y)(\gamma_n^3(z))'_z]\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&+\lambda\Delta t\sum_{p,q,t}\beta_{pqt}^{(n)}\int_a^b\int_c^d\int_e^f z^2[(\gamma_p^1(x))'_x\gamma_q^2(y)\gamma_t^3(z)(\gamma_k^1(x))'_x\gamma_m^2(y)\gamma_n^3(z)\\
&+\gamma_p^1(x)(\gamma_q^2(y))'_y\gamma_t^3(z)\gamma_k^1(x)(\gamma_m^2(y))'_y\gamma_n^3(z)+\gamma_p^1(x)\gamma_q^2(y)(\gamma_t^3(z))'_z\gamma_k^1(x)\gamma_m^2(y)(\gamma_n^3(z))'_z]\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&+\lambda\Delta t\sum_{p,q,t}\beta_{pqt}^{(n)}\int_a^b\int_c^d\int_e^f [(\gamma_p^1(x))'_x\gamma_q^2(y)\gamma_t^3(z)(\gamma_k^1(x))'_x\gamma_m^2(y)\gamma_n^3(z)\\
&+\gamma_p^1(x)(\gamma_q^2(y))'_y\gamma_t^3(z)\gamma_k^1(x)(\gamma_m^2(y))'_y\gamma_n^3(z)+\gamma_p^1(x)\gamma_q^2(y)(\gamma_t^3(z))'_z\gamma_k^1(x)\gamma_m^2(y)(\gamma_n^3(z))'_z]\mathrm{d}x\mathrm{d}y\mathrm{d}z
\end{aligned}
\tag{7.4}
$$

where

$$
\int_a^b \gamma_p^1(x)\gamma_k^1(x)\mathrm{d}x=\begin{cases}0, & k\le p-2\\ h/6, & k=p-1\\ 2h/3, & k=p\\ h/6, & k=p+1\\ 0, & k\ge p+2\end{cases}\qquad \int_a^b(\gamma_p^1(x))'_x(\gamma_k^1(x))'_x\mathrm{d}x=\begin{cases}0, & k\le p-2\\ -1/h, & k=p-1\\ 2/h, & k=p\\ -1/h, & k=p+1\\ 0, & k\ge p+2,\end{cases}
$$

and $\int_c^d\gamma_q^2(y)\gamma_m^2(y)\mathrm{d}y$, $\int_c^d(\gamma_q^2(y))'_y(\gamma_m^2(y))'_y\mathrm{d}y$, $\int_e^f\gamma_t^2(z)\gamma_n^2(z)\mathrm{d}z$, $\int_e^f(\gamma_t^2(z))'_z(\gamma_n^2(z))'_z\mathrm{d}z$ can also be written in the similar way.

If $N_x = N_y = N_z = 4$, then $h = 0.1$. Choosing $\lambda = 2.0$. Note that the problem in this paper is a nonlinear problem, $a(U^n)$, $b(U^n)$ and $f(U^n)$ in (3.3a) are the value of n-th level, and not the the value of $(n+1)$-th level, so that the time step Δt should be sufficiently small, thus we choose $\Delta t = 10^{-5}$. Fig. 1 shows the numerical solutions at $y = 0.3$, $t = 0.05$ and Table 1 gives the maximum absolute error, the average absolute error and the L^2 norm error.

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